Algebraic Cycles and Flag Varieties

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Abstract

This course is a general introduction to the theory of algebraic cycles and motives of twisted flag varieties.

We first recall basic concepts related to root systems, classification of linear algebraic groups and the associated projective homogeneous varieties. We then discuss twisted flag varieties – our main object of study, e.g., generalized Severi-Brauer varieties, anisotropic quadrics, versal flag varieties. As the next step, we introduce their Chow groups, and the associated category of Grothendieck-Chow motives. We provide several classical examples of motivic decompositions of twisted flag varieties. At the end of our course, we intend to demonstrate how a modern Schubert calculus technique (e.g. nil-Hecke modules) can be applied to obtain new/different proofs of motivic decompositions.

1 Root systems, root and weight lattices, Dynkin diagrams

Motivating example

We start with the following classical example.

Let $n \geq 1$ and consider the Euclidean space $\mathbb{R}^{n+1}$ with standard basis $\{e_0, \ldots, e_n\}$. Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a hyperplane consisting of vectors whose sum of coordinates equals zero

$$\mathcal{H} := \Big\{ \sum_{i=0}^{n} a_i e_i \colon \sum_{i=0}^{n} a_i = 0 \Big\}.$$

Let Φ denote the subset of $\mathcal{H}$ consisting of all differences of standard vectors

$$\Phi := \{e_i - e_j\}_{i \neq j}.$$

Clearly, $\Phi \cap m\Phi \neq \emptyset$ if and only if $m = \pm 1$. Observe that each vector from Φ can be uniquely expressed as a $\mathbb{Z}$-linear combination of vectors from the subset

$$\Delta := \{e_{i-1} - e_i\}_{i=1,\ldots,n}.$$

Moreover, if $\alpha \in \Phi$ and

$$\alpha = \sum_{i=1}^{n} c_i(e_{i-1} - e_i),$$

then either all coefficient $c_i \geq 0$, or all $c_i \leq 0$. In other words Φ splits into two disjoint subsets Φ^+ and Φ^-, where

$$\Phi^+ := \{e_i - e_j\}_{i<j} \quad \text{and} \quad \Phi^- := \{e_i - e_j\}_{i>j}.$$

Let Λ denote the $\mathbb{Z}$-linear span of Δ. Observe that Λ is a free $\mathbb{Z}$-module of rank n, and if taken with real coefficients it gives precisely $\mathcal{H}$.

For each vector $\alpha \in \Phi$, consider an orthogonal reflection s_α which fixes the hyperplane orthogonal to $\alpha = e_i - e_j$. It is given by the following formula

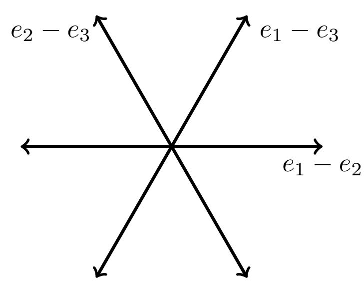
$$s_\alpha(\beta) := \beta - (\alpha, \beta)\,\alpha,$$

where $(\cdot,\cdot)$ denotes the usual dot-product in $\mathbb{R}^{n+1}$. Observe that s_α simply switches e_i and e_j (the ith and the jth coordinates) and, hence, it leaves both Φ and Λ invariant:

$$s_\alpha(\beta) \in \Phi \text{ (resp. } \Lambda), \text{ for all } \beta \in \Phi \text{ (resp. } \Lambda).$$

Let W be the group generated by all reflections s_α, $\alpha \in \Delta$ (the multiplication in W is given by the composite). Indeed, W consists of all permutations of standard vectors $\{e_0, \ldots, e_n\}$, hence, it can be identified with the symmetric group S_{n+1} on the set of indices $\{0, \ldots, n\}$. By definition, the group W leaves Φ and Λ invariant.

The introduced subset of vectors Φ provides an example of a *root system* of type A and of rank n. For $n = 2$ it can be diagrammatically represented in the hyperplane $\mathcal{H}$ as



Root systems

We generalize this example as follows. We now set Φ to be a finite non-empty subset of non-zero vectors in the Euclidean space $\mathbb{R}^n$ with an inner product $(\cdot,\cdot)$. For each vector $\alpha \in \Phi$ we define a linear transformation $s_\alpha \colon \mathbb{R}^n \to \mathbb{R}^n$ called a reflection by

$$s_\alpha(x) := x - \tfrac{2(\alpha,x)}{(\alpha,\alpha)}\,\alpha, \quad x \in \mathbb{R}^n.$$

We say Φ is a *root system* if the following conditions are satisfied:

(i) Vectors from Φ span the whole vector space $\mathbb{R}^n$,

(ii) If $m \in \mathbb{R}$ and $\Phi \cap m\Phi \neq \emptyset$, then $m = \pm 1$,

(iii) $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$,

(iv) $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

Elements of the set Φ are called *roots*.

Remark 1.1. Condition (ii) requires all our root systems to be reduced, and condition (iv) enforces all of them to be crystallographic.

We denote by Λ_r the $\mathbb{Z}$-linear span of vectors from Φ and called it the *root lattice*. One can show that there exists a subset $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of Φ such that it forms a $\mathbb{Z}$-basis of Λ_r and for each root $\alpha \in \Phi$ in the presentation

$$\alpha = \sum_{i=1}^{n} c_i \alpha_i$$

coefficients c_i are either all non-negative or all non-positive. In this way, the set Φ splits into a disjoint union $\Phi = \Phi^+ \amalg \Phi^-$ of subsets of *positive and negative roots* respectively. Elements of the subset Δ are called *simple roots* and $|\Delta| = n$ is called the *rank* of a root system. The group W generated by reflections s_α, $\alpha \in \Delta$ (called simple reflections) is called the *Weyl group* of the root system Φ.

To each root $\alpha \in \Phi$ we can associate a linear form

$$\alpha^\vee \colon \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \tfrac{2(\alpha,x)}{(\alpha,\alpha)}$$

called a *coroot*. We define the *Cartan matrix* $C = (c_{i,j})_{i,j}$ of the root system by setting the entries to be

$$c_{i,j} = \alpha_j^\vee(\alpha_i), \quad i,j = 1 \ldots n.$$

We define the ith *fundamental weight* ω_i to be a vector in $\mathbb{R}^n$ such that

$$\alpha_j^\vee(\omega_i) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

By definition, we obtain $\alpha_i = \sum_{j=1}^{n} c_{i,j}\omega_j$. So the rows of the Cartan matrix consist of coefficients expressing simple roots in terms of fundamental weights. We then define the *weight lattice* Λ_w of the root system Φ to be the $\mathbb{Z}$-linear span of the set of fundamental weights. It can also be described as

$$\Lambda_w = \{x \in \mathbb{R}^n \mid \alpha_i^\vee(x) \in \mathbb{Z} \text{ for each } i = 1 \ldots n\}.$$

By the very definition, $\Lambda_r \subseteq \Lambda_w$ and both lattices coincide if taken with rational coefficients

$$\Lambda_r \otimes_{\mathbb{Z}} \mathbb{Q} = \Lambda_w \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Moreover, its quotient group Λ_w/Λ_r is a finite abelian group called the *fundamental group* of the root system.

Example 1.2. For the root system of type A of rank 2, the Cartan matrix is given by

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and the fundamental weights are given by

$$\omega_1 = \tfrac{1}{3}(2\alpha_1 + \alpha_2), \qquad \omega_2 = \tfrac{1}{3}(\alpha_1 + 2\alpha_2).$$

The fundamental group $\Lambda_w/\Lambda_r = \{0, \omega_1, 2\omega_1 \equiv \omega_2\} \simeq \mathbb{Z}/3\mathbb{Z}$ is then the cyclic group of order 3.

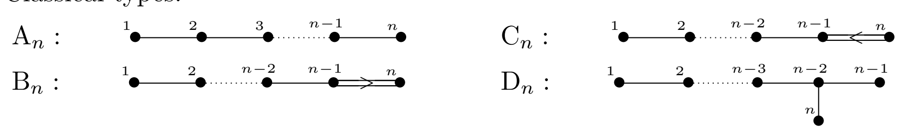
Classification

A root system is called *irreducible* if it cannot be represented as a direct sum of root systems, i.e., the root lattice Λ_r cannot be represented as $\Lambda_r = \Lambda_r' \oplus \Lambda_r''$, where $\Phi' \subset \Lambda_r'$ and $\Phi'' \subset \Lambda_r''$ are some root systems.

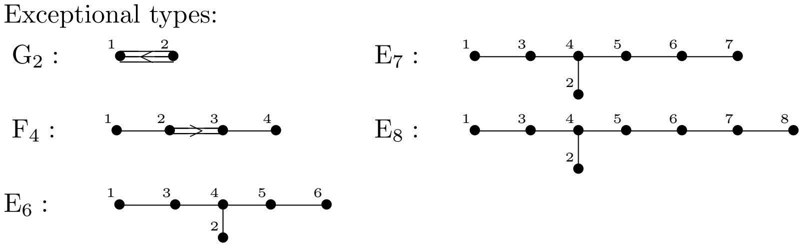
We associate a graph $\mathfrak{D}$ called the *Dynkin diagram* to any irreducible root system Φ. Its vertices correspond to the set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, and the number of edges connecting two different vertices α_i and α_j is given by the product $c_{i,j} \cdot c_{j,i}$ of coefficients of the Cartan matrix. Moreover, if there is more than one edge connecting α_i and α_j, we assign a direction to the edges between α_i and α_j as follows: if $c_{i,j} < c_{j,i}$, then α_j points to α_i. In this case, we observe that $(\alpha_i, \alpha_i) < (\alpha_j, \alpha_j)$. If there is a single edge, between α_i and α_j, then $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_j)$.

All irreducible root systems can be classified via their Dynkin diagrams. The latter consist of the following types (our enumeration of vertices follows Bourbaki; the lower index denotes the rank of a root system)

Classical types:



Exceptional types:



Recall that the Weyl group W of an irreducible root system is the group generated by all simple reflections s_α, $\alpha \in \Delta$. It can be shown that the group W has the following presentation

$$W = \langle s_1, \ldots, s_n \colon (s_i s_j)^{m_{i,j}} = 1 \rangle$$

where $s_i = s_{\alpha_i}$ denotes the generator corresponding to the simple root $\alpha_i \in \Delta$, and the exponents $m_{i,j}$ are determined as follows:

If $i = j$, then $m_{ii} := 1$. Otherwise, if $i \neq j$, it depends on the number of edges between α_i and α_j in the respective Dynkin diagram:

$$\begin{aligned}
m_{i,j} &:= 2 \text{ if } c_{i,j}c_{j,i} = 0 \text{ (no edge)},\\
m_{i,j} &:= 3 \text{ if } c_{i,j}c_{j,i} = 1 \text{ (single edge)},\\
m_{i,j} &:= 4 \text{ if } c_{i,j}c_{j,i} = 2 \text{ (double edge)},\\
m_{i,j} &:= 6 \text{ if } c_{i,j}c_{j,i} = 3 \text{ (triple edge)}.
\end{aligned}$$

The group W provides an example of the finite *Coxeter group*. By definition each element of W can be written as a product of generators (simple reflections)

$$w = s_{i_1} s_{i_2} \cdots s_{i_r}.$$

The minimal number of such generators required to express an element $w \in W$ is unique and is called the *length* of w. We denote the length by $\ell(w)$. Any expression of w as a product of $\ell(w)$ simple reflections is called a *reduced* expression (or a *reduced word*).

Exercise 1.3. Describe a root system of type D_4 (describe simple roots, positive and negative roots, fundamental weights). Show that the fundamental group Λ_w/Λ_r is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Exercise 1.4. Prove that $\det C = 2$, where C is the Cartan matrix of a root system of type C_n, $n \geq 2$.

References

[Hu90] J. Humphreys, *Reflection groups and Coxeter groups.* Cambridge studies in Advanced Math. 29, Cambridge Univ. Press 1990. xii+204 pp.

2 Nil-Hecke rings

Nil-Coxeter and nil-Hecke rings

Let W be a Coxeter group with a fixed generating set $\{s_1, \ldots, s_n\}$ subject to the Coxeter relations $(s_i s_j)^{m_{i,j}} = 1$ for $1 \leq i, j \leq n$. The *nil-Coxeter ring* $\mathcal{N}$ associated to W is the unital ring with generators $\{X_1, \ldots, X_n\}$ satisfying the *nilpotent relations*

$$X_i^2 = 0 \quad \text{for } 1 \leq i \leq n,$$

and *braid relations*

$$\underbrace{X_i X_j X_i \ldots}_{m_{i,j}} = \underbrace{X_j X_i X_j \ldots}_{m_{i,j}} \quad \text{for } i \neq j,\ 1 \leq i, j \leq n.$$

In other words, $\mathcal{N}$ is obtained from the integral group ring of the Coxeter group W by substituting the relation $s_i^2 = e$ with $X_i^2 = 0$.

Lemma 2.1. (a) *Let $\mathcal{N}$ be the nil-Coxeter ring associated to W and let $w \in W$. If $w = s_{i_1} s_{i_2} \ldots s_{i_r} = s_{j_1} s_{j_2} \ldots s_{j_r}$ are two reduced expressions for w, then*

$$X_{i_1} X_{i_2} \ldots X_{i_r} = X_{j_1} X_{j_2} \ldots X_{j_r}.$$

In particular, the product $X_{i_1} X_{i_2} \ldots X_{i_r}$ does not depend on the reduced expression of w and, hence, can be denoted X_w.

(b) *Furthermore, if $u, v \in W$, then*

$$X_u \cdot X_v = \begin{cases} X_{uv} & if \quad \ell(u) + \ell(v) = \ell(uv), \\ 0 & otherwise. \end{cases}$$

Exercise 2.2. * Show that if one replaces the nilpotent relations by the idempotent relations ($X_i^2 = X_i$) in the definition of the nil-Coxeter ring, then the statement (a) of the lemma still holds. Find an analogue of the formula (b) in this case.

Consider now root Λ_r and weight Λ_w lattices (free finitely generated abelian groups) of the root system Φ. Recall that $\Lambda_r \subset \Lambda_w$ and $rk_{\mathbb{Z}} \Lambda_r = rk_{\mathbb{Z}} \Lambda_w = n$ is the rank of Φ. Fix an intermediate W-invariant subgroup (an intermediate lattice) $\Lambda : \ \Lambda_r \subset \Lambda \subset \Lambda_w$.

Exercise 2.3. Prove or disprove: any intermediate subgroup $G : \Lambda_r \subset G \subset \Lambda_w$ is W-invariant.

Exercise 2.4. Describe all intermediate lattices for the root system of type D_4 (provide the respective bases as linear combinations of fundamental weights).

Choose a basis $\{x_1, \ldots, x_n\}$ of Λ, and consider the symmetric algebra

$$S = Sym_{\mathbb{Z}}(\Lambda).$$

As a ring S is isomorphic to the polynomial ring in variables $x_1, \ldots, x_n$. Consider the left S-module

$$\mathfrak{N} := S \otimes_{\mathbb{Z}} \mathcal{N}$$

obtained from the nil-Coxeter ring $\mathcal{N}$ by extending the scalars to polynomials. By Lemma **??** it is a free finitely generated left S-module with a basis $\{X_w\}_{w \in W}$.

The following relation (which we call an affine relation) tells us how to commute polynomials and generators X_i

$$X_i \, x_j = s_i(x_j) \, X_i + \alpha_i^\vee(x_j) \quad \text{for all } 1 \le i, j \le n. \tag{Aff}$$

Therefore, it turns the module $\mathfrak{N}$ into a ring with the usual juxtaposition multiplication. We call it the *nil-Hecke ring*. Observe that it depends on the root system Φ and the intermediate lattice Λ.

The relations (Aff) also induce the following relations in $\mathfrak{N}$:

Lemma 2.5. *For any $w \in W$ and $p \in \Lambda$ we have the following relation in $\mathfrak{N}$:*

$$X_w p = w(p) X_w + \sum_{w' \xrightarrow{\beta} w} \beta^\vee(p) X_{w'},$$

where $w' \xrightarrow{\beta} w$ means $w = w' s_\beta$ for some root $\beta \in \Phi^+$ with $\ell(w) = \ell(w') + 1$.

Remark 2.6. Labelled arrows $w' \to w$ define the *Bruhat graph*: a directed labelled graph where the vertex set is the set of elements of the Coxeter group W and the edge set consists of directed edges $w' \to w$ labelled by βs. Moreover, edges of the Bruhat graph give cover relations of the so-called (strong) Bruhat order: we say $w' < w$ if and only if there is a directed chain of edges starting at w' and pointing toward w.

Exercise 2.7. Draw the Bruhat graph for the root system of type A_2.

Proof. We proceed by induction on $\ell = \ell(w)$. If $\ell = 1$, then the proposition follows directly from (Aff). Now assume that the relation holds for all $u \in W$ with $\ell(u) < \ell$.

Consider $w = s_i u$ such that $\ell(w) = \ell(u) + 1$. By induction, we have

$$\begin{aligned} X_w p = X_i \cdot (X_u p) &= X_i \cdot \big(u(p) X_u + \sum_{u' \xrightarrow{\gamma} u} \gamma^\vee(p) X_{u'}\big) \\ &= \big(X_i u(p)\big) X_u + \sum_{u' \xrightarrow{\gamma} u} \big(X_i \gamma^\vee(p)\big) X_{u'} \\ &= w(p) X_w + \alpha_i^\vee(u(p)) X_u + \sum_{u' \xrightarrow{\gamma} u} \gamma^\vee(p)\, X_i \cdot X_{u'}. \end{aligned}$$

First note that $u \xrightarrow{u^{-1}(\alpha_i)} w$ and, hence the second summand belongs to the desired sum in the relation. Now suppose that $w' \xrightarrow{\beta} w$ and $u \neq w'$ and hence $w = w' s_\beta = s_i u$. This implies $u = (s_i w') s_\beta$ with $\ell(u) = \ell(s_i w') + 1$ and thus $s_i w' \xrightarrow{\beta} u$. Conversely, if $u' \xrightarrow{\gamma} u$, then $u' = s_i w'$ from some $w' \xrightarrow{\gamma} w$ if and only if $\ell(s_i u') = \ell(u') + 1$. The formula then follows. ☐

Example 2.8. Consider the Coxeter group of type A_3 and let $w = s_1 s_2 s_3$. Then for any $p \in \Lambda$ we have

$$X_w p = w(p) X_w + \alpha_3^\vee(p) X_{s_1 s_2} + s_3(\alpha_2)^\vee(p) X_{s_1 s_3} + s_3 s_2(\alpha_1)^\vee(p) X_{s_2 s_3}.$$

Remark 2.9. An embedding of intermediate lattices $\Lambda \hookrightarrow \Lambda'$ induces an embedding of the respective symmetric algebras, and, hence, polynomial rings $S \hookrightarrow S'$. This further induces an embedding of the respective nil-Hecke rings $\mathfrak{N} \hookrightarrow \mathfrak{N}'$. This embedding becomes an isomorphism after inverting the order$|\Lambda'/\Lambda|$. In other words, we obtain an isomorphism between $\mathfrak{N}$ and $\mathfrak{N}'$ if we extend the coefficient ring $\mathbb{Z}$ to $\mathbb{Z}[\frac{1}{|\Lambda'/\Lambda|}]$. Observe that the original rings $\mathfrak{N}$ and $\mathfrak{N}'$ are not necessarily isomorphic over $\mathbb{Z}$.

Example 2.10. Let Φ be the root system of type A_n. The Weyl group W is the permutation group generated by $\{s_1,\dots,s_n\}$ subject to the Coxeter relations:

$$s_i^2=1,\quad s_is_{i+1}s_i=s_{i+1}s_is_{i+1}\quad\text{and}\quad s_is_j=s_js_i\quad\text{if } |i-j|>1.$$

The nil-Coxeter group $\mathcal{N}$ is generated by $\langle 1,X_1,\dots,X_n\rangle$ with the analogous relations:

$$X_i^2=0,\quad X_iX_{i+1}X_i=X_{i+1}X_iX_{i+1},\quad\text{and}\quad X_iX_j=X_jX_i\quad\text{if } |i-j|>1. \tag{1}$$

Let $\Lambda=\Lambda_w$ be the weight lattice. The weight lattice has a basis of fundamental weights $\{\omega_1,\dots,\omega_n\}$. We can write $\omega_i=e_1+e_2+\dots+e_i$, where $\{e_1,\dots,e_{n+1}\}$ denotes the standard coordinate basis of $\mathbb{R}^{n+1}$. The nil-Hecke ring is then given by

$$\mathfrak{N}_w=\mathbb{Z}[\omega_1,\dots,\omega_n]\otimes_{\mathbb{Z}}\mathcal{N},$$

where $\mathcal{N}$ is the respective nil-Coxeter ring, and the affine relations (here we assume $\omega_0=\omega_{n+1}=0$) are

$$\begin{cases} X_i\omega_j=\omega_jX_i & \text{if } i\neq j,\\ X_i\omega_i=(\omega_{i-1}-\omega_i+\omega_{i+1})X_i+1 & \text{otherwise.}\end{cases} \tag{2}$$

Observe that $e_1=\omega_1$, $e_i=\omega_i-\omega_{i-1}$ for $i=2,\dots,n$, and $e_{n+1}=-\omega_n$ (here we assume $e_1+\ldots+e_{n+1}=0$). Substituting into (**??**) we obtain

$$X_ie_j=e_jX_i,\ \text{if } j\neq i,i+1,\ \text{ and } X_ie_i=e_{i+1}X_i+1,\ X_ie_{i+1}=e_iX_i-1$$

which can be rewritten as

$$X_ie_j-e_{s_i(j)}X_i=\delta_{i,j}-\delta_{i+1,j}. \tag{3}$$

If we substitute the polynomial ring $\mathbb{Z}[\omega_1,\dots,\omega_n]$ by $\mathbb{Z}[e_1,\dots,e_{n+1}]$ and forget the relation $e_1+\ldots+e_{n+1}=0$, we then obtain the classical definition of the *nil-Hecke algebra* generated by $\{e_1,\dots,e_{n+1},X_1,\dots,X_n\}$ which satisfies the nil-Coxeter relations given in (**??**) and the additional affine relations in (**??**).

Let $\Lambda=\Lambda_r$ be the root lattice instead of the weight lattice. The root lattice has a basis of simple roots $\Delta=\{\alpha_1,\dots,\alpha_n\}$, and in terms of the standard basis of $\mathbb{R}^{n+1}$ we have $\alpha_i=e_i-e_{i+1}$. The nil-Hecke ring is then

$$\mathfrak{N}_r=\mathbb{Z}[\alpha_1,\dots,\alpha_n]\otimes_{\mathbb{Z}}\mathcal{N}$$

with the additional affine relations:

$$\begin{cases} X_i\alpha_j=(\alpha_i+\alpha_j)X_i-1 & \text{if } |i-j|=1,\\ X_i\alpha_j=\alpha_jX_i & \text{if } |i-j|>1,\\ X_i\alpha_i=-\alpha_iX_i+2 & \text{otherwise.}\end{cases}$$

By definition $\mathfrak{N}_r$ is a proper subring of $\mathfrak{N}_w$. For example, take $n=1$. Then

$$\mathfrak{N}_r=\mathbb{Z}[\alpha]\Big\langle X\colon \substack{X^2=0,\\ X\alpha=-\alpha X+2}\Big\rangle\xrightarrow{\alpha\mapsto 2\omega}\mathbb{Z}[\omega]\Big\langle X\colon \substack{X^2=0,\\ X\omega=-\omega X+1}\Big\rangle=\mathfrak{N}_w.$$

Observe that the ring $\mathfrak{N}_r$ is not isomorphic to $\mathfrak{N}_w$ (over $\mathbb{Z}$), as they are not even isomorphic modulo 2 ($\mathfrak{N}_r$ is commutative, but $\mathfrak{N}_w$ is not). However, they become isomorphic after inverting 2.

Localization

In the definition of the nil-Hecke ring we can replace $\mathcal{N}$ by the group ring $\mathbb{Z}[W]$. Namely, consider the free left S-module

$$S_W := S \otimes_{\mathbb{Z}} \mathbb{Z}[W].$$

Each element of S_W can be written as a S-linear combination

$$\sum_{w\in W} p_w \delta_w,$$

where $\{\delta_w\}_{w\in W}$ is the standard basis of the group ring $\mathbb{Z}[W]$ and $p_w \in S$ are polynomial coefficients. For the identity element $e \in W$ we set $\delta_e = 1$. We define the ring multiplication on S_W to be the usual juxtaposition taken modulo the twisted commuting relation (degenerate affine relation):

$$w(p)\,\delta_w = \delta_w\, p, \quad p \in S,\ w \in W.$$

For example, if $p, q \in S$ and $w, v \in W$, then

$$p\,\delta_w \cdot q\,\delta_v = p\,w(q)\,\delta_{wv}.$$

The ring S_W is called the *twisted group algebra* associated to Φ and to Λ.

Let $Q := S[\frac{1}{\alpha}, \alpha \in \Phi]$ denote the localization of S at all $\alpha \in \Phi$. We define the *localized twisted group algebra* to be

$$Q_W := Q \otimes_{\mathbb{Z}} \mathbb{Z}[W]$$

with a similar twisted commuting relation. Observe that the canonical ring inclusion $S \hookrightarrow Q$ induces the ring inclusion $S_W \hookrightarrow Q_W$.

We now realize the nil-Coxeter and nil-Hecke rings as sub-rings of Q_W. For each simple root α_i consider an element

$$X_i := \tfrac{1}{\alpha_i}(\delta_e - \delta_{s_i}) \in Q_W.$$

We call it a *divided-difference element*. Consider a subring $\mathcal{N}'$ of Q_W generated by divided-difference elements $X_1, \ldots, X_n$. It can be shown that divided-difference elements satisfy nilpotent and braid relations.

Exercise 2.11. Show that divided-difference elements satisfy nilpotent and braid relations for type A_n (use properties of the symmetric group).

Hence, if $w = s_{i_1} \ldots s_{i_r}$ is a reduced expression, we can define

$$X_w := X_{i_1} \ldots X_{i_r}$$

with X_w being independent of choice of reduced expression of w.

Proposition 2.12. *The sub-ring $\mathcal{N}' \subseteq Q_W$ is isomorphic to the nil-Coxeter ring $\mathcal{N}$ of W.*

Proof. It suffices to show that the set $\{X_w\}_{w\in W}$ is linearly independent over S and is therefore a $\mathbb{Z}$-basis of $\mathcal{N}'$. Let $w \in W$ and fix a reduced word $w = s_{i_1}\cdots s_{i_r}$. Then expanding X_w in the $\{\delta_u\}_{u\in W}$ basis of Q_W yields:

$$\begin{aligned} X_w &= \tfrac{1}{\alpha_{i_1}}(\delta_e - \delta_{s_{i_1}})\cdots \tfrac{1}{\alpha_{i_r}}(\delta_e - \delta_{s_{i_r}}) \\ &= \tfrac{(-1)^{\ell(w)}}{\alpha_{i_1} s_{i_1}(\alpha_{i_2})\cdots(s_{i_1}s_{i_2}\cdots s_{i_{r-1}}(\alpha_{i_r}))}\,\delta_w + \sum_{u<w} c_{w,u}\,\delta_u. \end{aligned}$$

In particular, the transition matrix from $\{X_u\}_{u\in W}$ to $\{\delta_u\}_{u\in W}$ is upper triangular with respect to Bruhat order and is invertible over Q. Hence $\{X_u\}_{u\in W}$ is a Q-basis of Q_W which implies that $\{X_u\}_{u\in W}$ is S-linearly independent. $\square$

Note that the denominator of the coefficient $c_{w,w}$ in the proof above is equal to the product of roots in $\Phi^+ \cap w(\Phi^-)$. This set of roots is called the *inversion set* of $w \in W$. We now define

$$\mathfrak{N}' := S \otimes_{\mathbb{Z}} \mathcal{N}' \subset Q \otimes_Q Q_W = Q_W.$$

Proposition 2.13. *The sub-ring $\mathfrak{N}' \subseteq Q_W$ is isomorphic to the nil-Hecke ring of W.*

Proof. To prove that $\mathfrak{N}'$ is isomorphic to the nil-Hecke ring, it suffices to show that the relations (Aff) hold. Indeed, let $p \in \Lambda$. Then for each $1 \le i \le n$, we have

$$\begin{aligned} X_i p - s_i(p) X_i &= \tfrac{1}{\alpha_i}(\delta_e - \delta_{s_i})p - \tfrac{s_i(p)}{\alpha_i}(\delta_e - \delta_{s_i}) \\ &= \tfrac{p}{\alpha_i}\delta_e - \tfrac{s_i(p)}{\alpha_i}\delta_{s_i} - \tfrac{s_i(p)}{\alpha_i}\delta_e + \tfrac{s_i(p)}{\alpha_i}\delta_{s_i} = \alpha_i^\vee(x)\delta_e. \end{aligned}$$ $\square$

Polynomial representation

We define the $\circ$-action of the localized twisted group algebra Q_W on Q via

$$p\delta_w \circ q := pw(q). \tag{4}$$

Under this action the element X_i acts on Q as the classical divided-difference operator

$$X_i \circ q = \frac{q - s_i(q)}{\alpha_i}, \quad q \in Q.$$

Divided-difference operators preserve the polynomial subalgebra S, so the $\circ$-action gives a group homomorphism

$$Q_W \to \mathrm{Hom}_{\mathbb{Z}}(S, Q)$$

which restricts to a ring homomorphism

$$\rho\colon \mathfrak{N} \to \mathrm{End}_{\mathbb{Z}}(S).$$

Specifically, for any $y = \sum_{w\in W} a_w X_w \in \mathfrak{N}$ and $p \in S$, we have

$$\rho(y)(p) := \sum_{w\in W} a_w (X_w \circ p) \in S.$$

Lemma 2.14. (i) *If $p \in \Lambda$, then $X_i \circ p = \alpha_i^\vee(p)$.*

(ii) *The operator $X_i \circ$ - satisfies the twisted Leibniz rule:*

$$X_i \circ (pq) = (X_i \circ p)\, q + s_i(p)\, X_i \circ q, \quad p, q \in Q.$$

Proof. If $p \in \Lambda$, then

$$X_i \circ p = \tfrac{1}{\alpha_i}(p - (p - \alpha_i^\vee(p)\alpha_i)) = \alpha_i^\vee(p).$$

This proves part (i). For part (ii), we have

$$\begin{aligned} X_i \circ (pq) &= \tfrac{1}{\alpha_i}(pq - s_i(pq)) \\ &= \tfrac{1}{\alpha_i}(pq - s_i(p)q + s_i(p)q - s_i(pq)) \\ &= \tfrac{1}{\alpha_i}((p - s_i(p))q + s_i(p)(q - s_i(q))) \\ &= (X_i \circ p)\, q + s_i(p)\, X_i \circ q. \end{aligned}$$

□

Note that Lemma **??**.(i) implies that the affine relation (Aff) can be rewritten as

$$X_i\, p = s_i(p) X_i + X_i \circ p, \quad p \in \Lambda.$$

It is easy to see that $X_i \circ p = 0$ if and only if $s_i(p) = p$ (over $\mathbb{Z}$). Also for any homogeneous polynomial $p \in S$ we have

$$\deg(X_w \circ p) = \deg(p) - \ell(w)$$

and, hence, if $\deg(p) = \ell(w)$, then $X_w \circ p \in \mathbb{Z}$. This leads to the following

Lemma 2.15. *The representation ρ is faithful.*

Proof. Let w_0 be the element of maximal length in W (such an element is unique). Take a homogeneous polynomial $u_0 \in S$ of degree $\ell(w_0)$ such that $X_{w_0} \circ u_0 \in \mathbb{Z}$ is non-zero. Observe that $X_{w_0} \circ u_0 = 0$, if u_0 belongs to the ideal of W-invariants of S.

Suppose $y = \sum_w p_w X_w \in \mathfrak{N}$, $p_w \in S$ acts by 0 (i.e, $\rho(y) = 0$). Then for any $v \in W$,

$$0 = y \circ (X_v \circ u_0) = (y \cdot X_v) \circ u_0 = \sum_w p_w (X_w X_v \circ u_0).$$

Substituting v following the Bruhat graph backwards from $v = w_0$ to $v = e$, we obtain that $p_e = 0, \ldots, p_{w_0} = 0$. □

Exercise 2.16. Show that such u_0 (in the proof) always exists for a root system of type A.

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3 Chevalley groups and linear algebraic groups

Universal Chevalley group

Consider an irreducible root system Φ of rank ≥ 2 and a commutative unital ring R. Consider formal elements

$$x_\alpha(t),\ \alpha \in \Phi,\ t \in R.$$

If $t \in R^\times$ (t is invertible in R), define

$$w_\alpha(t) := x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t), \quad \text{and} \quad h_\alpha(t) := w_\alpha(t)w_\alpha(-1).$$

Define $\mathbb{G}_\Phi(R)$ to be a group generated by $x_\alpha(t)$ for all $\alpha \in \Phi$ and $t \in R$ subject to the relations:

(i) $x_\alpha(t)x_\alpha(u) = x_\alpha(t+u)$;

(ii) if $t, u \in R^\times$, then $h_\alpha(t)h_\alpha(u) = h_\alpha(tu)$;

(iii) if $\alpha + \beta \neq 0$, $\alpha, \beta \in \Phi$, then

$$[x_\alpha(t), x_\beta(u)] = \prod x_{i\alpha+j\beta}(N_{\alpha,\beta,i,j}t^i u^j),$$

where the product on the right-hand side is taken over all roots $i\alpha + j\beta$, $i, j > 0$ in a fixed order on Φ (the order tells us which formal element $x_\alpha(t)$ comes first in the commutator formula) and $N_{\alpha,\beta,i,j}$ are some given integer constants which do not depend on t and u but depend on that fixed order on Φ.

If Φ has rank 1 in the definition of the group $\mathbb{G}_\Phi(R)$ we replace the last relation (iii) by the following one:

$$w_\alpha(t)x_\alpha(u)w_\alpha(-t) = x_{-\alpha}(-t^2u), \quad \alpha \in \Phi,\ t \in R^\times,\ u \in R.$$

Example 3.1. Let $\Phi = \{\pm e_i \pm e_j, \pm e_k \mid 1 \leq i, j, k \leq n, i < j\}$ which is a root system of type B_n. Then the respective commutator relation (iii) in the definition of $\mathbb{G}_\Phi(R)$ can be described as follows. Let $\alpha, \beta \in \Phi$,

- If $0 \neq \alpha + \beta \notin \Phi$ then $[x_\alpha(t), x_\beta(u)] = 1$.

Otherwise, for integers $i < j$, $k < l$, and $a_1, a_2, a_3, a_4 = \pm 1$, we are in one of the following cases if $i \leq k$.

- $\alpha = a_1e_i + a_2e_j, \beta = a_3e_k + a_4e_l$ when

$$\begin{aligned} j = k :& [x_{a_1e_i+a_2e_j}(t), x_{-a_2e_j+a_4e_l}(u)] = x_{a_1e_i+a_4e_l}(-a_2tu) \\ i \neq k, j = l :& [x_{a_1e_i+a_2e_j}(t), x_{a_3e_k-a_2e_j}(u)] = x_{a_1e_i+a_3e_k}(-a_3tu) \\ i = k, j \neq l :& [x_{a_1e_i+a_2e_j}(t), x_{-a_1e_i+a_4e_l}(u)] = \begin{cases} x_{a_2e_j+a_4e_l}(a_2tu) & j < l \\ x_{a_4e_l+a_2e_j}(-a_4tu) & l < j. \end{cases} \end{aligned}$$

- $\alpha = a_1 e_i + a_2 e_j, \beta = a_3 e_k$ when

$$i = k : [x_{a_1e_i+a_2e_j}(t), x_{-a_1e_i}(u)] = x_{-a_1e_i+a_2e_j}(-tu^2)x_{a_2e_j}(a_2tu)$$
$$j = k : [x_{a_1e_i+a_2e_j}(t), x_{-a_2e_j}(u)] = x_{a_1e_i-a_2e_j}(tu^2)x_{a_1e_i}(-a_2tu)$$

- $\alpha = a_1 e_i, \beta = a_2 e_k$, then

$$[x_{a_1e_i}(t), x_{a_2e_k}(u)] = x_{a_1e_i+a_2e_k}(-2a_2tu).$$

If $k < i$ then the appropriate commutator is the inverse of one above.

The covariant functor
$$R \mapsto \mathbb{G}_\Phi(R)$$
(from unital commutative rings to groups) is called the (universal) Chevalley group associated to Φ. The constants $N_{\alpha,\beta,i,j}$ are called the *structure constants* of the Chevalley group. They can only take values ± 1, ± 2 or ± 3.

Set $N_{\alpha,\beta} = N_{\alpha,\beta,1,1}$. The constants $N_{\alpha,\beta}$ satisfy the following properties:

1. $N_{\alpha,\beta} = 0$ if $\alpha + \beta \notin \Phi$,
2. $N_{\alpha,\beta} = -N_{\beta,\alpha}$, and $N_{\alpha,-\beta} = N_{\beta,-\alpha}$,
3. If $-m\alpha + \beta \in \Phi$ but $-(m+1)\alpha + \beta \notin \Phi$, then
$$N_{\alpha,\beta} = \pm(m+1) \quad \text{and} \quad N_{\alpha,\beta}N_{-\alpha,-\beta} = -(m+1)^2,$$
4. If $\alpha + \beta + \gamma = 0$, $\alpha, \beta, \gamma \in \Phi$ then
$$N_{\alpha,\beta}/(\gamma,\gamma) = N_{\beta,\gamma}/(\alpha,\alpha) = N_{\gamma,\alpha}/(\beta,\beta)$$
5. For any $\alpha, \beta, \gamma \in \Phi$ (no pairs are opposite) we have the 2-cocycle equation:
$$N_{\beta,\gamma}N_{\alpha,\beta+\gamma} = N_{\alpha+\beta,\gamma}N_{\alpha,\beta}.$$

Remark 3.2. The constants $N_{\alpha,\beta}$ are precisely the structure constants of the complex Lie algebra associated to the root system Φ in the so-called Chevalley basis. For all root systems the constants $N_{\alpha,\beta,i,j}$ can be expressed in terms of the Lie constants.

Remark 3.3. A root system Φ is called simply laced if the respective Dynkin diagram does not have multiple edges. In this case at most one positive linear combination of α and β may be a root (their sum). So for simply laced root systems the only non-zero constants are the Lie constants $N_{\alpha,\beta} = N_{\alpha,\beta,1,1}$ (e.g., there is only one factor in the product).

Remark 3.4. If we skip the relation (ii) we obtain the definition of the Steinberg group associated to Φ.

Exercise 3.5. Given a root system of type A_n can one restore the coefficients $N_{\alpha,\beta}$ using only properties (1-5)? Justify your answer.

Subgroups and quotients of Chevalley groups

Let $\mathbb{G}_\Phi(R)$ be the universal Chevalley group corresponding to Φ. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be simple roots. Set $h_i(t) = h_{\alpha_i}(t)$ for $\alpha_i \in \Delta$ and $t \in R^\times$. Define the torus subgroup of $\mathbb{G}_\Phi(R)$

$$\mathbb{T}_\Phi(R) = \langle h_\alpha(t) \mid \alpha \in \Phi,\ t \in R^\times \rangle.$$

It is known that the group $\mathbb{T}_\Phi(R)$ is an abelian group generated by $h_i(t)$'s and each $h \in \mathbb{T}_\Phi(R)$ can be expressed uniquely as

$$h = \prod_{i=1}^{n} h_i(t_i), \text{ for some } t_i \in R^\times.$$

Moreover, the centre $Z(\mathbb{G}_\Phi(R))$ can be described as follows:

$$Z(\mathbb{G}_\Phi(R)) = \{\prod_{i=1}^{n} h_i(t_i) \in \mathbb{T}_\Phi(R) \mid \prod_{i=1}^{n} t_i^{\alpha_i^\vee(\beta)} = 1 \text{ for all } \beta \in \Delta\}.$$

Exercise 3.6. Prove the above facts concerning $\mathbb{T}_\Phi$ and the centre for the root system of type B_2 (use the example).

Example 3.7. Let $\mu_n \colon R \mapsto \mu(R) = \{t \in R^\times \mid t^n = 1\}$ be the functor of roots of unity. For a root system Φ of classical type A_n, B_n or C_n and any R we have

$$Z(\mathbb{G}_{\mathrm{A}_{n-1}}(R)) = \left\{ \prod_{i=1}^{n-1} h_i(t^i) \mid t \in \mu_n(R) \right\} \cong \mu_n(R)$$
$$Z(\mathbb{G}_{\mathrm{B}_n}(R)) = \{h_n(t) \mid t \in \mu_2(R)\} \cong \mu_2(R)$$
$$Z(\mathbb{G}_{\mathrm{C}_n}(R)) = \left\{ \prod_{\substack{i=1 \\ i \text{ odd}}}^{n} h_i(t) \mid t \in \mu_2(R) \right\} \cong \mu_2(R)$$

Type D_n depends on the parity of n. We have

$$Z(\mathbb{G}_{\mathrm{D}_n}(R)) = \left\{ \left(\prod_{\substack{i=1 \\ i \text{ odd}}}^{n-3} h_i(t_1 t_2) \right) h_{n-1}(t_1) h_n(t_2) \mid t_1, t_2 \in \mu_2(R) \right\} \cong (\mu_2 \times \mu_2)(R)$$

when n is even, and

$$Z(\mathbb{G}_{\mathrm{D}_n}(R)) = \left\{ \left(\prod_{\substack{i=1 \\ i \text{ odd}}}^{n-2} h_i(t^2) \right) h_{n-1}(t) h_n(t^{-1}) \mid t \in \mu_4(R) \right\} \cong \mu_4(R)$$

when n is odd.

Remark 3.8. If R is a field, then $Z(\mathbb{G}_\Phi(R))$ is finite.

We now introduce central quotients of Chevalley groups of classical types

A: Let m be an integer diving n. We define a quotient functor of type A_{n-1}

$$\mathbb{G}_{\mathrm{SL}_n/\mu_m}\colon R \mapsto \mathbb{G}_{\mathrm{A}_{n-1}}(R)/I$$

where I is the image of $\mu_m(R) \to \mu_n(R) \cong Z(\mathbb{G}_{\mathrm{A}_{n-1}}(R))$. When $m = n$ we denote this quotient functor by $\mathbb{G}_{\mathrm{PGL}_n}$.

B and C: We define quotient functors of types B_n and C_n as

$$\mathbb{G}_{\mathrm{SO}_{2n+1}}\colon R \mapsto \mathbb{G}_{\mathrm{B}_n}(R)/Z(\mathbb{G}_{\mathrm{B}_n}(R)) \text{ and } \mathbb{G}_{\mathrm{PSp}_{2n}}\colon R \mapsto \mathbb{G}_{\mathrm{C}_n}(R)/Z(\mathbb{G}_{\mathrm{C}_n}(R)).$$

D: We define several quotient functors of type D_n as follows: We first define the maximal quotient

$$\mathbb{G}_{\mathrm{PGO}_{2n}}\colon R \mapsto \mathbb{G}_{\mathrm{D}_n}(R)/Z(\mathbb{G}_{\mathrm{D}_n}(R)).$$

Then we define subquotients

$$\mathbb{G}_{\mathrm{SO}_{2n}}\colon R \mapsto \begin{cases} \mathbb{G}_{\mathrm{D}_n}(R)/I & n \text{ is even} \\ \mathbb{G}_{\mathrm{D}_n}(R)/J & n \text{ is odd,} \end{cases}$$

where when n is even, I is the image of the composition

$$\begin{aligned} \mu_2(R) &\to (\mu_2 \times \mu_2)(R) \cong Z(\mathbb{G}_{\mathrm{D}_n}(R)) \\ a &\mapsto (a, a) \end{aligned}$$

and when n is odd, J is the image of the composition

$$\mu_2(R) \to \mu_4(R) \cong Z(\mathbb{G}_{\mathrm{D}_n}(R)).$$

Finally, for an even n we define two extra subquotient functors

$$\mathbb{G}_{\mathrm{HSpin}^{\pm}_{2n}}\colon R \mapsto \mathbb{G}_{\mathrm{D}_n}(R)/I^{\pm}$$

where I^+ is the image of the composition

$$\begin{aligned} \mu_2(R) &\to (\mu_2 \times \mu_2)(R) \cong Z(\mathbb{G}_{\mathrm{D}_n}(R)) \\ a &\mapsto (a, 1), \end{aligned}$$

and I^- is the image of the composition

$$\begin{aligned} \mu_2(R) &\to (\mu_2 \times \mu_2)(R) \cong Z(\mathbb{G}_{\mathrm{D}_n}(R)) \\ a &\mapsto (1, a). \end{aligned}$$

To summarize we may depict all the central quotients as follows:

$$\begin{aligned}
\mathrm{A}_{n-1}\colon\ & \mathrm{SL}_n \longrightarrow \mathrm{SL}_n/\mu_m,\ m \mid n \longrightarrow \mathrm{PGL}_n\\
\mathrm{B}_n\colon\ & \mathrm{Spin}_{2n+1} \longrightarrow \mathrm{SO}_{2n+1} = \mathrm{PGO}_{2n+1}\\
\mathrm{C}_n\colon\ & \mathrm{Sp}_{2n} \longrightarrow \mathrm{PSp}_{2n}\\
\mathrm{D}_n\colon\ & \mathrm{Spin}_{2n} \longrightarrow \mathrm{SO}_{2n} \longrightarrow \mathrm{PGO}_{2n},\ \ \text{for } n \text{ odd}\\
\mathrm{D}_n\colon\ & \mathrm{Spin}_{2n} \longrightarrow \mathrm{SO}_{2n}\ \text{ or }\ \mathrm{HSpin}^{\pm}_{2n} \longrightarrow \mathrm{PGO}_{2n},\ \ \text{for } n \text{ even}
\end{aligned}$$

We call $\mathbb{G}_{\mathrm{SO}}, \mathbb{G}_{\mathrm{HSpin}}, \mathbb{G}_{\mathrm{PGO}}$, and $\mathbb{G}_{\mathrm{PSp}}$ the special orthogonal, half-spin, projective orthogonal, and projective symplectic Chevalley groups respectively. By the very definition all these groups are generated by formal symbols $x_\alpha(t)$ where relations in the definition of the universal Chevalley group together with the relations imposed by the central quotient form a complete set of relations. The smallest group of each type (maximal central quotients) are called the *adjoint Chevalley groups*.

Remark 3.9. In a similar way one can define quotients for Chevalley groups of exceptional types E_6 (by μ_3) and E_7 (by μ_2).

We define the (universal) unipotent subgroup of $\mathbb{G}_\Phi$ to be the functor

$$\mathbb{U}_\Phi\colon R \mapsto \mathbb{U}_\Phi(R) := \langle x_\alpha(t) \mid \alpha \in \Phi^+,\ t \in R\rangle.$$

We also define the Borel subgroup functor as

$$\mathbb{B}_\Phi\colon R \mapsto \mathbb{B}_\Phi(R) := \mathbb{T}_\Phi(R)\,\mathbb{U}_\Phi(R).$$

Chevalley groups are closely related to matrix groups

Example 3.10. There is an inclusion $\mathbb{G}_{\mathrm{A}_{n-1}}(R) \to \mathrm{SL}_n(R)$ given as follows Let $1 \le i < j \le n$ and let $t \in R$, then

$$\begin{aligned}
x_{e_i-e_j}(t) &\mapsto I_n + tE_{ij}\\
x_{-e_i+e_j}(t) &\mapsto I_n + tE_{ji}.
\end{aligned}$$

This map restricts to the maximal torus as

$$h_{e_i-e_j}(u) \mapsto diag(1,\ldots,\underset{i}{u},\ldots,\underset{j}{u^{-1}},\ldots,1),$$

the diagonal matrix with $u, u^{-1} \in R^\times$ in the i^{th} and j^{th} positions respectively and ones elsewhere. The Borel subgroup functor then will correspond to the subgroup of upper triangular matrices.

In a similar way one obtains inclusions of all the Chevalley groups into the respective matrix groups (we refer to [**?**] for concrete examples of classical types).

Connection to linear algebraic groups

We now discuss a connection between the Chevalley groups $\mathbb{G}_\Phi$, their subgroups and quotients on one side and split semisimple linear algebraic groups on the other side.

A linear algebraic group G can be viewed as a representable functor

$$G\colon Alg_F \to Groups \qquad R \mapsto Hom_{F\text{-}algebras}(H, R)$$

from the category of unital commutative F-algebras to the category of groups, represented by some commutative finitely generated Hopf algebra H over F such that the algebra $H \otimes_F \bar{F}$ is reduced, i.e., has no non-zero nilpotent elements (here $\bar{F}$ means the algebraic closure of F).

Using the language of affine schemes $G = Spec\, H$ is an affine algebraic variety over F.

Example 3.11.

- $\mathbb{G}_a$ the additive group. The respective Hopf algebra is $H = F[t]$.
- μ_n the group of n-th roots of unity. It is represented by $H = F[t]/(t^n - 1)$. This group is smooth iff $\operatorname{char} F$ does not divide n.
- GL_n the general linear group. The respective Hopf algebra
$$H = F[x_{ij}, \tfrac{1}{\det(x_{ij})}].$$
- $\mathbb{G}_m = \mathrm{GL}_1$ the multiplicative group.
- SL_n the special linear group. The respective Hopf algebra
$$H = F[x_{ij}]/(\det(x_{ij}) - 1).$$

A linear algebraic group G over F is called semisimple if it is non-trivial, connected and $G_{\bar{F}} = G \times_F \bar{F}$ does not have any non-trivial closed solvable connected normal subgroups. It is called (absolutely) simple if in addition $G_{\bar{F}}$ does not have any non-trivial closed connected normal subgroups.

One of the most important subgroups of a linear algebraic group G is a torus T. A linear algebraic group T over F is called a torus of rank n, if

$$T_{\bar{F}} \simeq \underbrace{\mathbb{G}_{m,\bar{F}} \times \ldots \times \mathbb{G}_{m,\bar{F}}}_{n}.$$

A torus T is called split, if such a presentation $T \simeq \mathbb{G}_m \times \ldots \times \mathbb{G}_m$ holds already over F.

Exercise 3.12. Consider the short exact sequence of groups

$$1 \to S^1 \to \mathbb{C}^\times \xrightarrow{z \mapsto z\bar{z}} \mathbb{R}^\times \to 1$$

It is known that S^1 is a group of $\mathbb{R}$-points of a non-split torus T over $\mathbb{R}$ of rank 1. Describe T as a representable functor over $\mathbb{R}$. Describe $T(R)$ for any $\mathbb{R}$-algebra R and describe the Hopf algebra H over $\mathbb{R}$.

A semisimple linear algebraic group G is called split of rank n, if its maximal subtorus (maximal with respect to inclusion) is split of rank n. If $F = \bar{F}$, then all semisimple linear algebraic groups over F are split.

Let G be a split simple simply-connected linear algebraic group over F. Assume that $char(F)$ does not divides its rank. Then one can show that there exists an irreducible root system Φ such that for any local strictly Henselian F-algebra R we have

$$G(R) = \mathbb{G}_\Phi(R).$$

In particular, $G(\bar{F}) = \mathbb{G}_\Phi(\bar{F})$.

Recall that a ring R is called a local strictly Henselian ring if

- R is local,
- The residue field $R/\mathfrak{m}$ is a separably closed field,
- If $f(x) \in R[x]$ is a polynomial whose image in $R/\mathfrak{m}[x]$ factors, then that factorization lifts to one in $R[x]$.

Local strictly Henselian rings are local rings in the étale topology. Since G is a sheaf in the étale site on Alg_F, the stalks of this sheaf are completely described by Chevalley generators in all cases except for the non-simply connected group of type A when $char(F)$ divides n (we will always exclude these cases). So for a split simple simply-connected G we may assume

$$G = \mathbb{G}_\Phi$$

after 'shififying' the latter. Under such identification

- The torus T corresponds to $\mathbb{T}_\Phi$ and the centre $Z(G)$ to $Z(\mathbb{G}_\Phi)$.
- The unipotent subgroup in G corresponding to a root α is the image of $u_\alpha\colon \mathbb{G}_a \hookrightarrow G$ such that $hu_\alpha(t)h^{-1} = u_\alpha(\alpha(h)t)$ for all $h \in T(F)$, $t \in F^\times$. This corresponds to the formal subgroup $\{x_\alpha(t) \mid t \in F\}$.
- The Borel subgroup B containing T (maximal closed connected solvable subgroup of G) corresponds to $\mathbb{B}_\Phi$.
- A split group isogeneous to a split simple simply-connected G corresponds to a central quotient of $\mathbb{G}_\Phi$.

  Recall that two linear algebraic groups G, G' over F are called (strictly) isogeneous if there is a linear algebraic group H and two surjective homomorphisms $\varphi\colon H \to G$ and $\varphi'\colon H \to G'$ with finite kernels such that for every F-algebra R the kernels of the homomorphisms $\varphi(R)$ and $\varphi'(R)$ are central in $H(R)$.

Remark 3.13. Following [**?**, A.§3.3(5)] for a field extension E/F we have

$$\mathbb{G}_\Phi(E) = [G(E), G(E)].$$

The group $G(E)$ is not always its own commutator subgroup.

So from now on whenever we deal with split simple simply-connected groups and their subgroups we will consider the respective Chevalley groups and their subgroups.

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4 Projective homogeneous varieties

We start with the following general definition. Let G be an algebraic group over a field F. A projective variety X over F is called a *homogeneous* G-variety if there is a morphism $\rho\colon G\times X\to X$ of algebraic varieties over F such that

$$\rho(g\cdot f,x)=g(f(x)) \text{ for all } f,g\in G(\bar F),\ x\in X(\bar F),$$

and the action of $G(\bar F)$ on $X(\bar F)$ is *transitive*, i.e. for every x, $y\in X(\bar F)$ there exists $g\in G(\bar F)$ such that $g\cdot x=y$, where $g\cdot x$ denotes $\rho(g,x)$. An $\bar F$-point x of a homogeneous G-variety X corresponds to the stabilizer subgroup G_x at x and for any two $\bar F$-points $x,y\in X$ the respective stabilizer subgroups are conjugate, i.e. we have $G_y=gG_xg^{-1}$, where $g\in G(\bar F)$ is such that $g\cdot x=y$. In other words, the conjugacy class of G_x describes $X(\bar F)$.

Let now G be a **split** semisimple simply-connected linear algebraic group over a field F. Let $\mathbb{G}_\Phi$ be the Chevalley group associated to G. So there is a root system Φ with a set of simple roots Δ. Let $J\subset\Delta$ be a subset of the set of simple roots and let Φ_J be a root subsystem generated by J. We then define the unipotent subgroup functor

$$\mathbb{U}_{\Phi,J}\colon R\mapsto \mathbb{U}_{\Phi,J}(R):=\langle x_\alpha(t)\mid \alpha\in\Phi^+\setminus\Phi_J,\ t\in R\rangle.$$

So $\mathbb{U}_{\Phi,\emptyset}=\mathbb{U}_\Phi$. We also set the Levi subgroup functor as

$$\mathbb{L}_{\Phi,J}\colon R\mapsto \mathbb{L}_{\Phi,J}(R)=\mathbb{T}_\Phi(R)\langle x_\alpha(t)\mid \alpha\in\Phi_J,\ t\in R\rangle.$$

We define the standard parabolic subgroup for J to be the semidirect product

$$\mathbb{P}_{\Phi,J}\colon R\mapsto \mathbb{P}_{\Phi,J}(R):=\mathbb{L}_{\Phi,J}(R)\,\mathbb{U}_{\Phi,J}(R),$$

where $\mathbb{L}_{\Phi,J}(R)$ acts on $\mathbb{U}_{\Phi,J}(R)$ by conjugation.

Remark 4.1. We can also look at the respective opposite subgroups

$$\mathbb{U}^-_{\Phi,J}\colon R \mapsto \mathbb{U}^-_{\Phi,J}(R) := \langle x_\alpha(t) \mid \alpha \in \Phi^- \setminus \Phi_J,\ t \in R\rangle,$$

and

$$\mathbb{P}^-_{\Phi,J}\colon R \mapsto \mathbb{P}_{\Phi,J}(R) := \mathbb{L}_{\Phi,J}(R)\,\mathbb{U}^-_{\Phi,J}(R).$$

We define the functor

$$\mathbb{G}_\Phi \,/\, \mathbb{P}_{\Phi,J}\colon R \mapsto \mathbb{G}_\Phi \,/\, \mathbb{P}_{\Phi,J}(R) := \{g\,\mathbb{P}_{\Phi,J}(R)g^{-1} \mid g \in \mathbb{G}_\Phi(R)\}.$$

One can show that it is representable by a projective homogeneous G-variety over F where stabilizer subgroups over $\bar{F}$ are conjugate to $\mathbb{P}_{\Phi,J}$ (the standard parabolic subgroup). In the sequel of this section we will denote such projective homogeneous varieties by G/P_J or simply by Φ/P_J, and call them split (projective homogeneous) varieties.

Example 4.2. We now provide examples of split projective homogeneous varieties (enumeration of roots follows Bourbaki). All these examples can be obtained by looking at the Chevalley subgroups in the respective matrix groups.

A_n: We have $\mathrm{A}_n/P_1 \simeq \mathrm{A}_n/P_n \simeq \mathbb{P}^n$, where $P_i = P_{\{\alpha_i\}}$ and, more generally,

$$\mathrm{A}_n/P_i \simeq \mathrm{A}_n/P_{n-i} \simeq \mathrm{Gr}(i, n+1),$$

where $\mathrm{Gr}(i, n+1)$ is the *Grassmannian* of i-dimensional linear subspaces in $\mathbb{A}^{n+1}$. The variety $\mathrm{A}_n/P_{\{\alpha_1,\alpha_n\}}$ is called the *incidence variety.* A point on this variety is given by the pair (l, H), where l is a line and H is a hyperplane in $\mathbb{A}^{n+1}$ such that $l \subset H$. In geometric terms, it is given by the equation $\sum_{i=0}^n x_i y_i = 0$ in $\mathbb{P}^n \times \mathbb{P}^n$, where x_i (resp. y_i) are the projective coordinates of the first (resp. second) factor.

Finally, $\mathrm{A}_n/P_\emptyset$ is isomorphic to the *variety of complete flags*: points are given by n-tuples of linear subspaces $(l_1, l_2, \ldots, l_n)$ in $\mathbb{A}^{n+1}$ such that $\dim l_i = i$ and $l_1 \subset l_2 \subset \ldots \subset l_n$.

B_n, D_n: The variety B_n/P_1 (resp. D_n/P_1) is isomorphic to a smooth projective *quadric* Q of dimension $2n-1$ (resp. $2n-2$) given by the equation $x_1^2 - x_2^2 + x_3^2 - \ldots = 0$, where $x_1, \ldots, x_{2n+1}$ (resp. $2n+1$ is replaced by $2n$) are the projective coordinates. The variety B_n/P_n (resp. D_n/P_n or D_n/P_{n-1}) is a (resp. a connected component of) *maximal orthogonal Grassmannian* that is a variety of maximal totally-isotropic linear subspaces in the quadratic space of rank $2n+1$ (resp. $2n$). The variety B_n/P_i, $i < n$ (resp D_n/P_i, $i < n-1$) is a Grassmannian of isotropic linear subspaces of dimension i.

G_2, F_4, E_6: The variety G_2/P_2 is isomorphic to a 5-dimensional smooth projective quadric and the variety G_2/P_1 is a 5-dimensional Fano variety. The variety E_6/P_6 is isomorphic to the so called *Cayley plane* $\mathbb{OP}^2$ that is the octonionic projective plane of dimension 16. The variety F_4/P_4 can be identified with a hyperplane section of E_6/P_6.

Cellular decompositions

Let X be a smooth projective variety over a field F. We call X a *cellular space* over F if there exists a finite decreasing filtration by closed proper subvarieties (not necessarily smooth)

$$X = X_0 \supsetneq X_1 \supsetneq \ldots \supsetneq X_n$$

such that each complement $X_i \smallsetminus X_{i+1}$ is isomorphic to a finite disjoint union of affine spaces $\mathbb{A}^{r_i}$ (called cells) over F of dimensions r_i.

Example 4.3. Consider the following descreasing filtration of $\mathbb{P}^n$ consisting of projective linear subspaces

$$\mathbb{P}^n \supset \mathbb{P}^{n-1} \supset \ldots \supset \mathbb{P}^1 \supset pt,$$

where on each step $\mathbb{P}^i \smallsetminus \mathbb{P}^{i-1} \simeq \mathbb{A}^i$.

Consider the following decompositions into a disjoint union of double cosets

$$\mathbb{G}_\Phi(R) = \mathbb{U}_\Phi(R) \setminus \mathbb{G}_\Phi(R) / \mathbb{U}_\Phi(R),$$

and

$$\mathbb{G}_\Phi(R) = \mathbb{B}_\Phi(R) \setminus \mathbb{G}_\Phi(R) / \mathbb{B}_\Phi(R),$$

It can be shown that for $R = F$ one can take representatives of the first to be elements of the normalizer subgroup

$$N_{\mathbb{T}_\Phi(F)}(\mathbb{G}_\Phi(F)) \subset \mathbb{G}_\Phi(F),$$

and representatives of the second to be elements of the Weyl group

$$W = N_{\mathbb{T}_\Phi(F)}(\mathbb{G}_\Phi(F)) / \mathbb{T}_\Phi(F).$$

For the respective algebraic groups it translates into the Bruhat decomposition over F:

$$G = \coprod_{w \in W} B\dot{w}B,$$

where $\dot{w}$ denotes a representative of $w \in W$ in $N_T(G)$.

Let J be a subset of Δ and let $W_J \subset W$ be the subgroup generated by all s_i with $\alpha_i \in \Delta$. Let $W^J \subset W$ denote the set of *representatives of minimal length* in the set W/W_J. Observe that for each coset there exists only one such representative. Moreover, one can show that

$$W^J = \{\, w \in W \mid \forall\, \alpha_i \in J,\ \ell(ws_i) = \ell(w) + 1 \,\}.$$

We introduce a partial order '$\leq$' on the set W^J as follows: we say that $w \leq w'$ for $w, w' \in W^J$ if some (or any) reduced word for w' contains some reduced word for w as a substring. This is an order induced by the Bruhat order.

Let X be the projective homogeneous G-variety of dimension d corresponding to a subset $J \subset \Phi$. By the Bruhat decomposition we have

$$X = \coprod_{w \in W^J} B\dot{w} \cdot x,$$

where x is an F-point on X corresponding to the stabilizer subgroup P_J, and $X'_w = B\dot{w}x \simeq \mathbb{A}^{l(w)}$ is the affine cell. The closure of the respective cell $X_w = \overline{X'_w}$ is called the *Schubert variety*. We have

$$X_{w'} = \coprod_{w \leq w'} X'_w \text{ and } \quad X_w \subseteq X_{w'} \text{ if and only if } w \leq w'.$$

Observe that the Schubert varieties X_w are non-smooth in most of the cases.

The Bruhat decomposition defines a cellular filtration on X where

$$X_i := \bigcup_{w \in W^J,\, l(w) \leq d-i} X_w, \quad i = 0 \dots d.$$

We have $X = X_0$, $X_i \smallsetminus X_{i-1} \simeq \coprod_{w \in W^J,\, l(w)=d-i} \mathbb{A}^{d-i}$ and $X_d = x$.

In particular, the variety X contains exactly $|\{w \in W^J \mid l(w) = r\}|$ cells of dimension r.

Example 4.4. Let Q be a projective quadric given by the equation

$$q(x_{-n}, \dots, x_{-1}, x_0, x_1, \dots, x_n) = x_0^2 + x_1 x_{-1} + \ldots + x_n x_{-n} = 0$$

(a split quadric). Then the Bruhat decomposition gives the following cellular decomposition of Q:

$$Q \supset Q \cap \{x_{-1} = 0\} \supset \ldots \supset Q \cap \{x_{-n+1} = \ldots = x_{-1} = 0\}$$
$$\supset \mathbb{P}^{n-1} \supset \mathbb{P}^{n-2} \supset \ldots \supset pt.$$

Now we generalize the notion of a cellular filtration.

Let X be a smooth projective variety over F. Suppose that there is a filtration by closed subvarieties

$$X = X_n \supsetneq X_{n-1} \supsetneq \ldots \supsetneq X_0 \supsetneq X_{-1} = \emptyset$$

and for each $i = 0, \dots, n$ there are smooth projective varieties Y_i and maps $X_i \setminus X_{i-1} \to Y_i$ making $X_i \setminus X_{i-1}$ an affine fibration over Y_i. Then X is called relative cellular. The varieties Y_i are a part of the structure.

A cellular filtration is also a relative cellular filtration, where the varieties Y_i are disjoint unions of points.

Example 4.5. Consider the general form of an isotropic quadratic form q over a field F with $\operatorname{char} F \neq 2$ (a quadratic form q is isotropic if the equation $q = 0$

has a non-trivial solution, otherwise, q is called anisotropic) and the respective projective quadric

$$Q = \{[x : y : \ldots : z_1 : \ldots : z_n] \mid xy + q'(z_1, \ldots, z_n) = 0\}.$$

Consider the filtration

$$Q \supset X \supset pt,$$

where $pt = [1 : 0 : \ldots : 0]$ and $X = \{y = 0\}$. Define the morphism

$$X \setminus pt \to Q' = \{q' = 0\} \quad \text{as } [x : y : z_1 : \ldots z_n] \mapsto [z_1 : \ldots : z_n].$$

Exercise 4.6. Show that $Q \setminus X \simeq \mathbb{A}^n$ and the morphism $X \setminus pt \to Q'$ is an $\mathbb{A}^1$-fibration.

Thus, an isotropic quadric is a relative cellular space.

Example 4.7. Let $q = \sum_{i=1}^{n+1} x_i x_{-i}$ be a $(2n+2)$-dimensional split quadratic form. Let Q be the respective projective quadric. Consider the inclusion

$$Q \supset \mathbb{P}^n = \{x_{-1} = x_{-2} = \ldots = x_{-n-1} = 0\} \supset \emptyset$$

with the map $Q \setminus \mathbb{P}^n \to \mathbb{P}^n$ defined on the points as

$$[x_{-n-1} : \ldots : x_{-1} : x_1 : \ldots : x_{n+1}] \mapsto [x_1 : \ldots : x_{n+1}].$$

This map is an $\mathbb{A}^n$-fibration and, thus, we have another relative cellular structure on Q: $Q \supset \mathbb{P}^n \supset pt$.

Twisted flag varieties

We fix a root system Φ, a subset $J \subset \Delta$, the Chevalley group $\mathbb{G}_\Phi$ over F and a split projective homogeneous variety $\mathbb{G}_\Phi / \mathbb{P}_{\Phi,J} = \Phi/P_J$.

We say that an algebraic variety X is a twisted form of Φ/P_J if

$$X \times_F \bar{F} \simeq \Phi/P_J \times_F \bar{F}.$$

The descent theory gives an isomorphism of pointed sets between the set of isomorphisms classes of twisted forms of $\mathbb{G}_\Phi / \mathbb{P}_{\Phi,J}$ and the first Galois cohomology set over F

$$\mathrm{Twist}_F(\mathbb{G}_\Phi / \mathbb{P}_{\Phi,J}) \simeq \mathrm{H}^1_{\mathrm{Gal}}(F, \mathrm{Aut}(\mathbb{G}_\Phi / \mathbb{P}_{\Phi,J})),$$

or equivalently, for a split G we can translate it as

$$\mathrm{Twist}_F(G/P_J) \simeq \mathrm{H}^1_{\mathrm{Gal}}(F, \mathrm{Aut}(G/P_J)),$$

Suppose $G = G^{ad}$ is adjoint. There is an exact sequence of algebraic groups

$$1 \to G \xrightarrow{Inn} \mathrm{Aut}(G) \xrightarrow{\pi} \Gamma \to 1,$$

where the first arrow is given by taking inner automorphisms $Inn(g)(x) = gxg^{-1}$ and the group Γ is finite. Let $\mathrm{Aut}(G; J)$ denote the open subgroup of $\mathrm{Aut}(G)$ which fixes the respective conjugacy class of P_J. It is the inverse image of some subgroup of Γ.

Exercise 4.8. Show that the identity component $\mathrm{Aut}(G;J)^o$ can be identified with G.

If G is simple, except of few cases there is an isomorphism

$$\phi\colon \mathrm{Aut}(G;J) \simeq \mathrm{Aut}(G/P_J)$$

so we can relate twisted forms of G/P_J to the twisted forms of the group G itself $\mathrm{H}^1_{\mathrm{Gal}}(F,\mathrm{Aut}(G))$ and to the classes of $\mathrm{H}^1_{\mathrm{Gal}}(F,G)$ which classify the so-called G-torsors:

Given an arbitrary algebraic group G over F, a G-variety Y over F is called a G-torsor (principal G-bundle) if the map

$$Y \times_F G \longrightarrow Y \times_F Y \text{ defined on points by } (y,g) \mapsto (y,yg)$$

is an isomorphism of varieties over F. Now let Y be a G-torsor. Choose a $\bar{F}$-rational point $x \in Y(\bar{F})$. Let $\gamma \in \mathrm{Gal}(\bar{F}/F)$ be an element in the absolute Galois group. Then there exists a unique element $z(\gamma) \in G(\bar{F})$ such that $\gamma(x) = x\cdot z(\gamma)$. The constructed map $z\colon \mathrm{Gal}(\bar{F}/F) \to G(\bar{F})$ defines a 1-cocycle which does not depend on the choice of x.

Example 4.9. The group G itself gives a trivial G-torsor.

Consider a Galois extension E/F with the Galois group G. Then $Y = \mathit{Spec}\, E$ provides an example of G-torsor over F.

Example 4.10. Embed $G \subset GL_n$ as a closed subgroup. Then $GL_n \to GL_n/G$ is a principal G-bundle. Consider its generic fiber $Y \to \mathit{Spec}\, K$, where K is the function field of GL_n/G. This gives the so-called generic G-torsor over K.

A homomorphism $H \to G$ of algebraic groups over F induces a map

$$\mathrm{H}^1_{\mathrm{Gal}}(F,H) \to \mathrm{H}^1_{\mathrm{Gal}}(F,G).$$

So for a split adjoint simple G the map $\mathit{Inn}\colon G \to \mathrm{Aut}(G)$ induces the map

$$\mathrm{H}^1_{\mathrm{Gal}}(F,G) \to \mathrm{H}^1_{\mathrm{Gal}}(F,\mathrm{Aut}(G))$$

The twisted forms of G which belong to the image of this map are called the *inner twisted forms* of G. So if there are no non-trivial automorphisms of the Dynkin diagram of G (types B_n, C_n, E_7, E_8, F_4, G_2), then all twisted forms of G are inner.

Example 4.11. We now provide some examples of twisted flag varieties. In the cases of quadrics (type B) we assume $\mathrm{char}\, F \neq 2$, in the case of E_6 we assume that $\mathrm{char}\, F \neq 2,3$.

- The twisted forms of $\mathbb{P}^n$ are Severi-Brauer varieties $\mathrm{SB}(A)$, where A is a central simple algebra over F of degree $\deg A = n+1$. A Severi-Brauer variety $\mathrm{SB}(A)$ is the variety of the right ideals of A of reduced dimension 1, i.e., of dimension $n+1$. All twisted forms of $\mathbb{P}^n$ are of this kind.

The image of

$$\mathrm{H}^1_{\mathrm{Gal}}(F, \mathrm{SL}_{n+1} / \mu_m) \to \mathrm{H}^1_{\mathrm{Gal}}(F, \mathrm{PGL}_{n+1})$$

with $m \mid n+1$ classifies the central simple F-algebras of exponent dividing m.

- The twisted forms of odd-dimensional smooth quadrics are again quadrics. The situation with even-dimensional quadrics over F turns out to be more sophisticated: The class of twisted forms of a $2n$-dimensional split quadric consists of involutive varieties. The latter are the varieties of the right ideals I of a central simple algebra A over F of degree $2n$ with an orthogonal involution σ of the first kind such that $\sigma(I)I = 0$ and $\dim I = 2n$. If the algebra A splits, we get a quadric. The inner twisted forms of a split quadric correspond to the case when the discriminant of σ is trivial.

  From the point of view of Galois cohomology, consider a n-dimensional split quadratic space (V, q) over F. Then the orthogonal group O_n is the group of isometries of (V, q). The (pointed) set $\mathrm{H}^1_{\mathrm{Gal}}(F, \mathrm{O}_n)$ classifies the isomorphism classes of non-singular n-dimensional quadratic forms. The image of

$$\mathrm{H}^1_{\mathrm{Gal}}(F, \mathrm{SO}_n) \to \mathrm{H}^1_{\mathrm{Gal}}(F, \mathrm{O}_n)$$

  classifies the quadratic forms with trivial discriminant, and the image of

$$\mathrm{H}^1_{\mathrm{Gal}}(F, \mathrm{Spin}_n) \to \mathrm{H}^1_{\mathrm{Gal}}(F, \mathrm{O}_n)$$

  classifies the quadratic forms with trivial discriminant and trivial Clifford invariant.

- The twisted forms of the Cayley plane $\mathbb{OP}^2 = \mathrm{E}_6/P_1$ are the varieties of singular lines in an Albert algebra over F. There is an interesting subclass of these varieties which corresponds to Albert algebras of the so called *first Tits construction*. This algebras have a property that they become split over some cubic field extension. The cohomology classes corresponding to the Albert algebras of the first Tits construction belong to the image of

$$\mathrm{H}^1_{\mathrm{Gal}}(F, \mathrm{PGL}_3 \times \mu_3) \to \mathrm{H}^1_{\mathrm{Gal}}(F, \mathrm{F}_4).$$

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5 Chow Groups and Chow motives

We consider a field F and the category Var_F of algebraic varieties over F. We introduce the Chow groups in several steps:

Definition of the Chow group

$Z_i(X)$: For an integer $i \geq 0$ and a scheme X we define *the groups of cycles* $Z_i(X)$ to be the free abelian group generated by the irreducible subvarieties of X of dimension i. Set $Z_*(X) = \bigoplus_{i=0}^{\dim X} Z_i(X)$.

$ord_x(f)$: Let $W \subset X$ be an irreducible subvariety of X of dimension $i+1$, let $f \in F(W)^\times$ be a rational function on W and let $x \in W$ be a point such that $\dim \overline{\{x\}} = i$. Then the order of f at x is defined to be the integer

$$ord_x(f) = length_{\mathcal{O}_{W,x}}(\mathcal{O}_{W,x}/g\mathcal{O}_{W,x}) - length_{\mathcal{O}_{W,x}}(\mathcal{O}_{W,x}/h\mathcal{O}_{W,x})$$

where $f = g/h$ for $g, h \in \mathcal{O}_{W,x}$.

$\mathrm{div}(f)$: With hypothesis as above we define a cycle $[\mathrm{div}(f)]$ on X to be

$$[\mathrm{div}(f)] = \sum_{\substack{\dim \overline{\{x\}} = i \\ x \in W}} ord_x(f) \left[\overline{\{x\}}\right]$$

where $[V]$ denotes the class of an irreducible closed subvariety $V \subset X$ in $Z_*(X)$.

$Rat_i(X)$: The group $Rat_i(X) \subset Z_i(X)$ is the subgroup generated by:

$$Rat_i(X) = \left\langle [\mathrm{div}(f)] \mid f \in k(W)^\times, \dim W = i+1 \right\rangle.$$

$\mathrm{CH}_i(X)$: The i-th Chow group of X is the i-th group of cycles modulo rational equivalence:

$$\mathrm{CH}_i(X) = Z_i(X)/Rat_i(X).$$

$\mathrm{CH}_*(X)$: The Chow group of X is the direct sum

$$\mathrm{CH}_*(X) = \bigoplus_{i=0}^{\dim X} \mathrm{CH}_i(X).$$

Example 5.1. Let $X = \mathrm{Spec}\, F$. Then

$$\mathrm{CH}_0(X) = \mathbb{Z},$$
$$\mathrm{CH}_i(X) = 0 \text{ for } i \geq 1$$

Let $X = \operatorname{Spec} A$ for a Dedekind domain A. Then

$$\begin{aligned} &\mathrm{CH}_0(X) = \text{Class group}(A),\\ &\mathrm{CH}_1(X) = \mathbb{Z},\\ &\mathrm{CH}_i(X) = 0 \text{ for } i \geq 2.\end{aligned}$$

For a commutative ring R one can define the Chow groups with coefficients in R as $\mathrm{CH}_i(X, R) := \mathrm{CH}_i(X) \otimes_{\mathbb{Z}} R$. For example, one can take $R = \mathbb{Z}/p$ or $R = \mathbb{Q}$.

Properties of the Chow groups

We now list several basic properties of the Chow groups.

Proper push-forward. Let $f : X \to Y$ be a proper morphism. By the definition of a proper morphism, if $V \subset X$ is a subvariety of X, then so is $f(V)$. So we obtain a map $Z_i(X) \to \mathrm{CH}_i(Y)$ defined by:

$$f_* : [V] \mapsto \begin{cases} 0, & \dim f(V) < \dim V;\\ [F(V) : F(f(V))] \cdot [f(V)], & \dim f(V) = \dim V.\end{cases}$$

This passes to rational equivalence to define a map:

$$f_* : \ \mathrm{CH}_i(X) \to \mathrm{CH}_i(Y)$$

Example 5.2. Consider the proper structural map $\pi : X \to \operatorname{Spec} F$. Then there is a map

$$\deg = \pi_* : \ \mathrm{CH}_0(X) \to \mathrm{CH}_0(\operatorname{Spec} F) = \mathbb{Z}$$

called *the degree map.*

If X is a smooth projective quadric then

$$\operatorname{Im} \pi_* = \begin{cases} \mathbb{Z}, & X \text{ has rational points};\\ 2\mathbb{Z}, & \text{otherwise,}\end{cases}$$

and $\operatorname{Ker} \pi_* = 0$.

Flat pull-back. Let $f : X \to Y$ be a flat morphism of constant relative dimension n. We define a map on the cycle level by

$$f^* : [W] \mapsto [f^{-1}(W)]$$

for $[W] \in Z_*(Y)$. This gives a well-defined map

$$f^* : \ \mathrm{CH}_i(Y) \to \mathrm{CH}_{i+n}(X).$$

For a smooth variety X set

$$\mathrm{CH}^i(X) = \mathrm{CH}_{\dim X - i}(X).$$

Let Sm_F denote the category of smooth varieties over F. Then one can extend the definition of the pull-back and show that for any $f : X \to Y$ in Sm_F there is a pull-back map

$$f^* : \ \mathrm{CH}^i(Y) \to \mathrm{CH}^i(X).$$

Homotopy invariance. For an affine fibration $\pi\colon V \to X$ in Var_F of constant dimension n the pull-back $\pi^*\colon \mathrm{CH}_i(X) \to \mathrm{CH}_{i+n}(V)$ is an isomorphism.

Localization. Let U be an open subvariety of X and let $Z = X \setminus U$ be its complement. Let $i\colon Z \to X$ and $j\colon U \to X$ be inclusions. Then for all m there is an exact sequence

$$\mathrm{CH}_m(Z) \xrightarrow{i_*} \mathrm{CH}_m(X) \xrightarrow{j^*} \mathrm{CH}_m(U) \to 0.$$

Example 5.3. Given an absolute cellular variety $X \in \mathrm{Var}_F$ with cellular filtration

$$X = X_n \supset X_{n-1} \supset X_{n-2} \supset \ldots \supset X_0 \supset \emptyset$$

where $X_i \setminus X_{i-1} = \bigcup_j U_{ij}$ where U_{ij} are isomorphic to affine spaces then the abelian group $\mathrm{CH}_*(X)$ is generated by the closures $[\overline{U_{ij}}] \in \mathrm{CH}_*(X)$.

Intersection product for smooth varieties. Let $X \in \mathrm{Sm}_F$ and let $\Delta_X\colon X \to X \times X$ denote the diagonal map. We have

$$\begin{array}{c} \mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathrm{CH}^j(X) \to \mathrm{CH}^{i+j}(X \times X) \xrightarrow{\Delta_X^*} \mathrm{CH}^{i+j}(X) \\ [V] \otimes [W] \mapsto [V \times W] \end{array}$$

which defines the structure of a commutative graded ring on the Chow group $\mathrm{CH}^*(X)$.

Projection formula. If $f\colon X \to Y$ is proper in Sm_F then we have maps

$$\mathrm{CH}^*(Y) \xrightarrow{f^*} \mathrm{CH}^*(X) \xrightarrow{f_*} \mathrm{CH}^*(Y)$$

and for $a \in \mathrm{CH}^*(Y)$, $b \in \mathrm{CH}^*(X)$ we have

$$f_*(f^*(a) \cdot b) = a \cdot f_*(b).$$

Equivalently, we may say that f_* is a homomorphism of $\mathrm{CH}^*(Y)$-modules.

Chern classes. For a vector bundle $\mathcal{E} \to X$ over $X \in \mathrm{Var}_k$ and for any $i \geq 0$ there is a class $c_i(\mathcal{E}) \in \mathrm{CH}_{\dim X - i}(X)$ called the i-th Chern class of $\mathcal{E}$. The Chern classes satisfy the following properties:

- $c_0(\mathcal{E}) = 1_X$ (for an irreducible X), $c_i(\mathcal{E}) = 0$ for $i > rk_X\mathcal{E}$;
- If $\mathcal{L}$ is a line bundle over X and V is an m-dimensional subvariety of X, then $\mathcal{L}|_V$ is isomorphic to $\mathcal{O}_V(C)$ for some Cartier divisor C of V. As for the respective Weil divisor we have

$$[C] = c_1(\mathcal{L}) \cdot [V] \in \mathrm{CH}_{m-1}(X).$$

- *Whitney sum.* If there is an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

  of vector bundles over X then

$$c(\mathcal{E};t) = c(\mathcal{E}';t) \cdot c(\mathcal{E}'';t),$$

  where $c(\mathcal{E};t) = \sum_{i\geq 0} c_i(\mathcal{E})t^i$ is *the Chern polynomial.*

Projective bundle formula. Let $\mathcal{E}$ be a locally free sheaf over X of rank $r+1$. The projective bundle of $\mathcal{E}$ is

$$\mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{E}^\vee)),$$

where $\mathcal{E}^\vee$ is the dual to $\mathcal{E}$. Then the following projective bundle theorem holds:

$$\mathrm{CH}_*(\mathbb{P}(\mathcal{E})) \simeq \bigoplus_{i=0}^{r} c_1(\mathcal{O}_{\mathcal{E}}(1))^i \pi^*(\mathrm{CH}_{*-r+i}(X)),$$

where $\pi\colon \mathbb{P}(\mathcal{E}) \to X$ is the projection.

Moreover, let $\xi = c_1(\mathcal{O}_{\mathcal{E}}(1))$. Then $\xi^{r+1} + c_1\xi^r + c_2\xi^{r-1} + \ldots + c_{r+1} = 0$, where $c_i = c_i(\pi^*(\mathcal{E}))$.

Example 5.4. Let $X = \mathrm{Spec}\, F$ and $\mathcal{E} = \mathbb{A}^{r+1}$ the affine space of dimension $r+1$. Then $\mathbb{P}(\mathcal{E}) = \mathbb{P}^r$ and $c_i(\mathcal{O}(1)) = h^i$, where h is a hyperplane section of $\mathbb{P}^r$. Projective bundle theorem says that there is a ring isomorphism

$$\mathrm{CH}^*(\mathbb{P}^r) \simeq \mathbb{Z}[\xi]/(\xi^{r+1}).$$

Correspondences

Let SmPr_F denote the category of smooth projective algebraic varieties over a field F. A correspondence $\alpha\colon X \to Y$ for $X, Y \in \mathrm{SmPr}_F$ is an element of the Chow group of $\mathrm{CH}_*(X \times Y) = \bigoplus_{i\geq 0} \mathrm{CH}_i(X \times Y)$.

Example 5.5. For a morphism $f\colon X \to Y$ in SmPr_F its graph gives the (associated) correspondence $[\Gamma_f] \in \mathrm{CH}_{\dim X}(X \times Y)$.

Let $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$ be correspondences. Define the composition of correspondences as

$$\beta \circ \alpha = p_{13*}\left(p_{12}^*(\alpha) \cdot p_{23}^*(\beta)\right) \in \mathrm{CH}^*(X \times Z)$$

where

$$\begin{array}{ccccc} & & X \times Y \times Z & & \\ & \swarrow^{p_{12}} & \downarrow^{p_{13}} & \searrow^{p_{23}} & \\ X \times Y & & X \times Z & & Y \times Z \end{array}$$

are projections on the respective factors.

Remark 5.6. Equivalently, we could define it via

$$\beta \circ \alpha = p_{13*}(\mathrm{id}_X \times \Delta_Y \times \mathrm{id}_Z)^*(\alpha \times \beta)$$

where

$$\begin{array}{ccc} X \times Y \times Z & \xrightarrow{\mathrm{id}_X \times \Delta_Y \times \mathrm{id}_Z} & X \times Y \times Y \times Z \\ {\scriptstyle \Delta_{X\times Y\times Z}}\downarrow & \nearrow & \\ (X \times Y \times Z) \times (X \times Y \times Z) & & \end{array}$$

Product of correspondences has the following properties:

Exercise 5.7. Prove that the element $\Delta_X := \Delta_{X*}([X]) \in \mathrm{CH}_*(X \times X)$ is the identity, i.e., $\Delta_X \circ \alpha = \alpha$ and $\beta \circ \Delta_X = \beta$ for all $\alpha\colon W \to X$ and $\beta\colon X \to Y$.

- The product of correspondences is associative, i.e., for $\alpha\colon W \to X$, $\beta\colon X \to Y$ and $\gamma\colon Y \to Z$ we have $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$.

We also have the following operations:

- For a correspondence $\alpha\colon X \to Y$

$$\begin{array}{rcl} \alpha^*\colon \mathrm{CH}(Y \times -) & \to & \mathrm{CH}(X \times -) \\ \beta & \mapsto & \beta \circ \alpha \end{array}$$

- For a correspondence $\beta\colon Y \to Z$

$$\begin{array}{rcl} \beta_*\colon \mathrm{CH}(- \times Y) & \to & \mathrm{CH}(- \times Z) \\ \alpha & \mapsto & \beta \circ \alpha \end{array}$$

- We have $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$ and $(\alpha \circ \beta)_* = \alpha_* \circ \beta_*$.
- If $g\colon Y \to Z$ is a morphism in $SmPr_k$ and $\alpha\colon X \to Y$ is a correspondence then $[\Gamma_g] \circ \alpha = (\mathrm{id} \times g)_*(\alpha)$.
- If $f\colon X \to Y$ is a morphism in $SmPr_k$ and $\beta\colon Y \to Z$ is a correspondence then $\beta \circ [\Gamma_f] = (f \times \mathrm{id})^*(\beta)$.
- Let

$$\begin{array}{rcl} \tau\colon X \times Y & \to & Y \times X \\ (x, y) & \mapsto & (y, x) \end{array}$$

  and $\alpha\colon X \to Y$ be a correspondence, then $\alpha^t := \tau_*(\alpha)\colon Y \to X$ is the transpose of α. Clearly $(\alpha^t)^t = \alpha$.

Exercise 5.8. Prove that if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms in SmPr_F, then $[\Gamma_{g\circ f}] = [\Gamma_g] \circ [\Gamma_f]$.

The category of correspondences Corr_F has as objects smooth projective varieties over a field F and morphisms are $\mathrm{Hom}(X,Y) = \mathrm{CH}_*(X \times Y)$. This is an additive category with addition given by $X \oplus Y := X \coprod Y$ (disjoint union) and has a tensor product given by $X \otimes Y := X \times Y$. The Yoneda lemma for this category is called the Manin Identity Principle:

In the above notation, if $\alpha\colon X \to Y$ in Corr_F then

- α is an isomorphism if and only if
- $\alpha^*\colon \mathrm{CH}(Y \times -) \to \mathrm{CH}(X \times -)$ is an isomorphism for all $-$'s if and only if
- $\alpha_*\colon \mathrm{CH}(- \times X) \to \mathrm{CH}(- \times Y)$ is an isomorphism for all $-$'s.

The category of correspondences of degree 0, denoted by Corr^0_F is defined as follows: objects are smooth projective varieties, and the morphisms are given by

$$\mathrm{Hom}(X,Y) = \coprod_{\ell} \mathrm{CH}_{\dim X_\ell}(X_\ell \times Y)$$

where the X_ℓ are the irreducible components of X. The category Corr^0_F is an additive category with tensor product and there is a functor

$$\begin{array}{rcl} \mathrm{Corr}^0_F & \to & \mathrm{Corr}_F \\ X & \mapsto & X \\ \mathrm{Hom}(X,Y) \ni f & \mapsto & f \in \mathrm{CH}_*(X \times Y). \end{array}$$

The category of Chow motives

Given an additive category Add_F, define the idempotent completion $\widetilde{\mathrm{Add}}_F$ to be the category whose objects are (A,p) where $A \in \mathrm{Add}_F$, $p \in \mathrm{Hom}(A,A)$ is a projector (that is, $p \circ p = p$) and morphisms are

$$\mathrm{Hom}((A,p),(B,q)) = q \circ \mathrm{Hom}(A,B) \circ p.$$

This category has the property that if p is a projector then

$$(A, \mathrm{id}_A) \simeq (A,p) \oplus (A, \mathrm{id}_A - p).$$

The category of Chow motives Mot_F is defined as the idempotent completion $\widetilde{\mathrm{Corr}}^0_F$ of the category of degree 0 correspondences Corr^0_F.

Example 5.9. Two important objects in the category Mot_F are:

- The motive of a point $\mathbb{Z} = (\operatorname{Spec} F, \mathrm{id})$;
- Consider the projective line $\mathbb{P}^1$ and $\mathrm{Hom}(\mathbb{P}^1, \mathbb{P}^1)$. The group $\mathrm{CH}^*(\mathbb{P}^1 \times \mathbb{P}^1)$ is a free abelian group generated by

$$\begin{array}{ll} p_0 := [pt \times \mathbb{P}^1], & [pt \times pt], \\ p_1 := [\mathbb{P}^1 \times pt], & [\mathbb{P}^1 \times \mathbb{P}^1]. \end{array}$$

We have $p_1 \circ p_1 = p_1$, $p_0 \circ p_0 = p_0$ and $p_1 + p_0 = \mathrm{id}_{\mathbb{P}^1}$. So we obtain a direct sum decomposition

$$(\mathbb{P}^1, p_1) \oplus (\mathbb{P}^1, p_0) \simeq (\mathbb{P}^1, \mathrm{id}_{\mathbb{P}^1})$$

in Mot_F. The motive $(\mathbb{P}^1, p_0)$ is called *the Tate motive* and is denoted by $\mathbb{Z}(1)$. We also have $(\mathbb{P}^1, p_1) \simeq \mathbb{Z}$.

For a motive M in Mot_F we define its Tate twist (or shift) as

$$M \otimes \mathbb{Z}(i) = M \otimes (\mathbb{Z}(1))^{\otimes i} = M(i).$$

If X is a smooth projective variety over F then the object $M(X) := (X, \Delta_X)$ in M_F is called the Chow motive of X. Sometimes we call the Tate twists $\mathbb{Z}(i)$ the Tate motives.

Let $X \in \mathrm{SmPr}_F$ be a relative cellular scheme. That is, there is a filtration

$$X = X_n \supset X_{n-1} \supset \ldots \supset X_0$$

such that $X_i \subset X$ are closed and the $U_i = X_i \setminus X_{i-1}$ are affine fibrations of constant rank d_i over some $Y_i \in \mathrm{SmPr}_F$. Then we have

$$M(X) \simeq \bigoplus_{i=0}^{n} M(Y_i)(d_i). \tag{5}$$

Example 5.10. The cellular filtration of $\mathbb{P}^n$ implies

$$M(\mathbb{P}^n) \simeq \bigoplus_{i=0}^{n} \mathbb{Z}(i).$$

More generally, if X is a split projective homogeneous variety of type J then

$$M(X) \simeq \bigoplus_{w \in W^J} \mathbb{Z}(l(w)).$$

The covariant functor CH_* from the category SmPr_F to the category of graded abelian groups factors through Mot_F. Namely, the following triangle commutes:

$$\begin{array}{ccc} \mathrm{SmPr}_F & \xrightarrow{\mathrm{CH}_*} & \mathrm{GrAb} \\ & \searrow \quad \nearrow_{\mathcal{R}_{\mathrm{CH}_*}} & \\ & \mathrm{Mot}_F & \end{array}$$

where for $(X, p) \in \mathrm{Mot}_F$ the realization functor $\mathcal{R}_{\mathrm{CH}_*}$ is given by $\mathcal{R}_{\mathrm{CH}_*}((X,p)) = p_{2*}(p \cdot p_1^*(\mathrm{CH}_*(X)))$ where $p_i \colon X \times X \to X$ is the i-th projection.

Therefore the motivic decomposition induced by the relative cellular filtration gives rise to a decomposition of Chow groups:

$$\mathrm{CH}_*(X) \simeq \bigoplus_{i=0}^{n} \mathrm{CH}_{*-d_i}(Y_i).$$

Equivariant Chow groups and motives

By the following process (the Borel construction) one can produce the equivariant analogue of Chow groups:

Let G be a linear algebraic group over a field F. Consider a system of G-representations V_i and its open subsets $U_i \subseteq V_i$ such that

- G acts freely on U_i and the quotient U_i/G exists as a variety over F,
- $V_{i+1} = V_i \oplus W_i$ for some representation W_i,
- $U_i \subseteq U_i \oplus W_i \subseteq U_{i+1}$, and $U_i \oplus W_i \to U_{i+1}$ is an open inclusion, and
- $codim(V_i \setminus U_i)$ strictly increases as i does and $\lim_i \dim V_i = \infty$.

Such a system exists and it is called a good system of representations of G.

Now let X be a smooth G-variety (e.g. a projective homogeneous variety). The inverse limit induced by pull-backs

$$\varprojlim_i \mathrm{CH}^*(U_i \times^G X), \quad U_i \times^G X = (U_i \times X)/G,$$

does not depend on the choice of the system (V_i, U_i) and defines the equivariant Chow group $\mathrm{CH}^*_G(X)$.

We then can define the category of G-correspondences $G\text{-}\mathrm{Corr}_F$ by starting from the category of smooth projective G-varieties $G\text{-}\mathrm{SmPr}_F$ with G-equivariant morphisms, and replacing correspondences $\mathrm{CH}(X \times Y)$ by G-equivaraint correspondences $\mathrm{CH}_G(X \times Y)$. The properties of the Borel construction guarantee that the product of G-correspondences is well-defined. Similarly, one can then look at the category of pairs (X, p), where p is a G-equivaraint projection, and define the category of G-equivariant motives. One can then show that any relative G-equivariant cellular filtration on X induces the direct sum decomposition of the respective G-equivariant motive of X. So if $\mathbb{P}^1 = \mathbb{A} \amalg \{pt\}$ is G-cellular, then there is the G-equivariant Tate motive.

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6 Motives of twisted flag varieties

Let G be a (not necessarily split) linear algebraic group over F which is an inner twisted form of a split simple linear algebraic group $\mathbb{G}_\Phi$ over F corresponding to a root system Φ. Let S be a maximal split torus of G, and let T be a maximal torus of G defined over F and containing S. We choose compatible orderings in the character groups S^* and T^*. Let Δ be the set of simple roots of $\mathbb{G}_\Phi$ with respect to $\mathbb{T}_\Phi$ so that $\mathbb{T}_\Phi \times_F \bar{F} = T \times_F \bar{F}$ (and the chosen ordering) and let Δ_0 be the subsystem of those roots which vanish on S.

Tits diagrams

We consider a data consisting of the Dynkin diagram of Φ and the subset $\Delta_0 \subset \Delta$. We represent this data graphically. Namely, we circle the vertices of the Dynkin diagram of Φ which do not belong to Δ_0. We call this diagram the *Tits diagram* of G. It does not depend on the choice of S and on the isogeny class of G. Thus, the group G is split if and only if all vertices in its Tits diagram are circled. On the other hand, the group G contains no (non-trivial) split tori iff no vertices in its Tits diagram are circled.

An equivalent definition of a Tits diagram in our situation is as follows:

A vertex i in the Dynkin diagram is circled iff the maximal parabolic subgroup $P_i = P_{\Delta\setminus\{i\}}$ of G is defined over F.

Example 6.1. All Tits diagrams have been classified

- A_n: The simple groups of type A_n which are inner twisted forms of SL_{n+1} are classified by the central simple algebras A of degree $n+1$. Namely, these are the groups $\mathrm{SL}_1(A)$ of elements of A of reduced norm 1.

  Let $d = \operatorname{ind} A$ and $r = \frac{n+1}{d} - 1$. One can see that the circled vertices in the Tits diagram of $\mathrm{SL}_1(A)$ are the vertices $\{d, 2d, \ldots, rd\}$ (enumeration of roots follows Bourbaki).

- B_n: The simply connected groups of type B_n are classified by regular $(2n+1)$-dimensional quadratic spaces (V, q) over F. Namely, these are the groups $\mathrm{Spin}(V, q)$.

  The circled vertices in the Tits diagram of $\mathrm{Spin}(V, q)$ (or $\mathrm{SO}(V, q)$) are the vertices $\{1, 2, \ldots, r\}$, where r is the Witt index of the quadratic form q.

- C_n: The simply connected groups of type C_n are classified by central simple algebras (A, σ) with symplectic involutions. Namely, these are the groups $\mathrm{Sp}(A, \sigma)$.

  The circled vertices in the respective Tits diagrams are $d, 2d, \ldots, rd$, where d is the index of A and r is the index of σ.

- D_n: The simply connected groups of inner type D_n are classified by central simple algebras A with a quadratic pair (σ, f) of trivial discriminant. Namely, these are the groups $\mathrm{Spin}(A, \sigma, f)$.

The circled vertices are $\{d, 2d, \ldots, rd\}$, where d is the index of A and r is the index of (σ, f).

For exceptional groups we indicate the Dynkin types of the Tits indices Δ_0:

E_6: Possible types of Δ_0 are $\emptyset$, D_4, A_2+A_2, and E_6.

E_7: Possible types of Δ_0 are $\emptyset$, D_4, D_6, E_6, $A_1+A_1+A_1$ (with $\Delta_0=\{2,5,7\}$), D_4+A_1, D_5+A_1, and E_7.

E_8: Possible types of Δ_0 are $\emptyset$, D_4, D_6, E_6, D_7, E_7, and E_8.

F_4: Possible types of Δ_0 are $\emptyset$, B_3, and F_4.

G_2: Possible types of Δ_0 are $\emptyset$ and G_2.

Relative cellular filtrations on twisted flags

Recall that by the Bruhat decomposition the projective homogeneous variety $\mathbb{G}_\Phi / \mathbb{P}_{\Phi,J}$ is a disjoint union of affice spaces (cells) and that these cells correspond to the coset representatives W^J, i.e., to the vertices of the respective Hasse diagram of W/W_J. Moreover, the dimension of the cell corresponding to $w \in W^J$ equals $\ell(w)$.

Now we generalize this situation to twisted flag varieties. Let G be an inner twisted form of a split simple linear algebraic group $\mathbb{G}_\Phi$ over F. So it corresponds to a $\mathbb{G}_\Phi$-torsor Y and there is $\Delta_0 \subset \Delta$. Let X be the respective twisted form (by Y) of $\mathbb{G}_\Phi / \mathbb{P}_{\Phi,J}$. So X is a projective homogeneous variety under the action of G. We construct a relative cellular filtration on X as follows:

Consider the set E of the representatives of minimal length in the set of double cosets $W_{\Delta_0}\backslash W/W_J$ (one can show that for each double coset such representative is unique). Let $m=|E|$. Let G_{Δ_0} be the subgroup of G generated by $X_{\pm\alpha}$ with $\alpha \in \Delta_0$. We can choose a bijection φ between $\{0,\ldots,m-1\}$ and E so that there is a relative cellular filtration on X

$$X = X_{m-1} \supset X_{m-2} \supset \ldots \supset X_0$$

such that for each $i=0,\ldots,m-1$ the variety $X_i \setminus X_{i-1}$ is an $\mathbb{A}^{l(\varphi(i))}$-fibration over Z_i, where Z_i is a projective G_{Δ_0}-homogeneous variety of type $\{\alpha \in \Delta_0 \mid \varphi(i)^{-1}\alpha \in \Phi_J\}$.

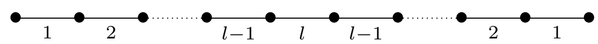
One can give a graphical interpretation of the varieites Z_i using Hasse diagrams as follows. Consider the Hasse diagram of $\mathbb{G}_\Phi / \mathbb{P}_{\Phi,J}$ and erase all the edges in it having labels in the set $\Delta \setminus \Delta_0$. The Hasse diagram splits then into several disjoint components. These components turn out to be the Hasse diagrams of the varieties Z_i over $\bar{F}$ and the ranks $l(\varphi(i))$ equal the lengths of the shortest words in the respective components.

Remark 6.2. The constructed filtration is interesting only if G is isotropic, i.e., if $\Delta_0 \neq \Delta$. In other words, if G contains a non-trivial split torus. In the case $\Delta_0 = \emptyset$ (i.e., if G is split), we recover the absolute filtration coming from the Bruhat decomposition.

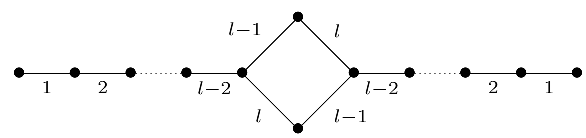
Example 6.3. In this example we review the Rost filtration of an isotropic quadric.

Let Q be an isotropic projective quadric of dimension n and let G be the respective orthogonal group (of type $\mathrm{D}_{(n+2)/2}$ if n is even or of type $\mathrm{B}_{(n+1)/2}$ if n is odd) acting on it. Since $Q \times_F \bar{F} \simeq (G \times_F \bar{F})/P_1$ and Q has an F-rational point (it is isotropic), the parabolic subgroup P_1 is defined over F.

Set $l = [(n+2)/2]$, take $\Delta_0 = J = \Delta \setminus \{1\}$ and consider the Hasse diagram for Q:



or



To visualized the Rost filtration following the described algorithm we have to erase all edges in the Hasse diagram labelled by 1. The Hasse diagram then splits into 3 pieces: 2 alone-standing vertices on the left and on the right and the middle part. An easy computation shows that this middle part is the Hasse diagram for a quadric Q' of dimension $n-2$.

Motives of generic flags

Recall that a G-variety Y is a G-torsor if there is an isomorphism

$$\phi\colon Y \times G \to Y \times Y, \quad (y,g) \mapsto (y, yg).$$

Consider the composite of the inverse ϕ^{-1} and the projection

$$Y \times Y \to Y \times G \to G.$$

This is a $P \times P$-equivariant map, where $(y_1,y_2)(b_1,b_2) = (y_1b_1, y_2b_2)$ and $g(b_1,b_2) = b_1^{-1}gb_2$. So there is an induced pull-back

$$\mathrm{CH}^*_G(G/P \times G/P) = \mathrm{CH}^*_{P\times P}(G) \to \mathrm{CH}^*_{P\times P}(Y \times Y) = \mathrm{CH}^*(Y/P \times Y/P),$$

where Y/P is the respective twisted flag variety. (Here we use the property of the equivariant Chow group that says that for any G-torsor $Y \to Y/G$, $\mathrm{CH}_G(Y) = \mathrm{CH}(Y/G)$.) Or equivalently, it induces the map

$$\gamma\colon End_{G\text{-}\mathrm{Mot}_F}(M(G/P)) \to End_{\mathrm{Mot}_F}(M(Y/P)).$$

We have the following key result:

Theorem 6.4. *If $G = \mathbb{G}_\Phi$ is split simple over a field F of characteristic 0, $P_J = \mathbb{P}_{\Phi,J}$, and if Y is a generic $\mathbb{G}_\Phi$-torsor over F (from the example), then γ induces a bijection between isomorphism classes of direct sum decompositions of $M(G/P_J)$ in G-Mot_F and direct sum decompositions of $M(Y/P_J)$ in Mot_F. Or equivalently, γ lifts idempotents and isomorphisms between idempotents.*

In particular, $M(Y/P_J)$ is indecomposable in Mot_F iff $M(G/P_J)$ is indecomposable in G-Mot_F.

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7 Nil-Hecke modules and equivariant motives

The W-action on motives

We fix a split simple group $G = \mathbb{G}_\Phi$ and its split maximal torus $T = \mathbb{T}_\Phi$. Let W be the Weyl group of G with respect to T and let X be an arbitrary smooth G-variety. There is a natural (left) action of W on $\mathrm{CH}_T(X)$. It can be realized by pull-backs induced by the right action of W on each step of the Borel construction $U \times^T X$ via

$$(u, x)T \cdot \sigma T = (u\sigma, \sigma^{-1}x)T, \quad \sigma \in N_G(T),$$

where U is taken to have a right G-action. By definition, the forgetful map $res^G_T \colon \mathrm{CH}_G(X) \to \mathrm{CH}_T(X)$ then factors through the W-invariants $\mathrm{CH}_T(X)^W$, so we obtain a map

$$res^G_T \colon \mathrm{CH}_G(X) \to \mathrm{CH}_T(X)^W.$$

We now replace G-equivariant correspondences $\mathrm{CH}_G(X \times Y)$ in the definition of the category of G-correspondences (G-motives) by correspondences from the image of the forgetful map

$$res^G_T \colon \mathrm{CH}_G(X \times Y) \to \mathrm{CH}_T(X \times Y)^W.$$

In this way we obtain the category of relative $G|T$-motives $G|T\text{-Mot}_F$. Or if we just replace it by the W-equivariant correspondences $\mathrm{CH}_T(X \times Y)^W$, we obtain the category of W-equivariant T-motives $T\text{-Mot}_F^W$.

There is a chain of functors

$$G\text{-Mot}_F \twoheadrightarrow G|T\text{-Mot}_F \hookrightarrow T\text{-Mot}_F^W \hookrightarrow T\text{-Mot}_F,$$

where the first functor is full and the remaining functors are faithful.

We have the following corollary of the theorem:

Corollary 7.1. *Let $P_J = \mathbb{P}_{\Phi,J}$ be the standard parabolic subgroup of G. There is a bijection between isomorphism classes of direct sum decompositions of $M(G/P_J)$ in G-Mot$_F$ and direct sum decompositions of $M(G/P_J)$ in $G|T$-Mot$_F$.*

On the other side there is a ring inclusion

$$End_{G|T\text{-}\mathrm{Mot}_F}(M(G/P_J)) \hookrightarrow End_{T\text{-}\mathrm{Mot}_F^W}(M(G/P_J))$$

Hence, if $M(G/P_J)$ is indecomposable in $T\text{-}\mathrm{Mot}_F^W$, then so is $M(Y/P_J)$ in Mot_F for a generic torsor Y. This leads to a natural question:

Can we explicitly describe the endomorphism ring $End_{T\text{-}\mathrm{Mot}_F^W}(M(G/P_J))$?

The W-equivariant endomorphisms.

By the very definition the endomorphism ring $End_{T\text{-}\mathrm{Mot}_F^W}(M(G/P_J))$ is

$$End(\mathrm{CH}_T(G/P_J \times G/P_J)^W, \circ),$$

where the composition is given by the correspondence product. Consider the realization functor $T\text{-}\mathrm{Mot}_F^W \to \mathrm{GrRings}$ induced by $M(X) \mapsto \mathrm{CH}_T(X)$. It identifies our endomorphism (correspondence) α with the W-equivariant endomorphism

$$\alpha_* \in End_{\mathrm{CH}_T(pt)}(\mathrm{CH}_T(G/P_J)).$$

We are going to describe the latter endomorphism ring using the Weyl action of the nil-Hecke ring $\mathfrak{N}$ of the root system Φ on its parabolic dual.

Let $S = Sym(T^*)$ be the polynomial ring from the definition of $\mathfrak{N}$ and let Q be its localization. Consider the free Q-module Q_{W/W_J} on the basis $\{\delta_w\}$ indexed by coset representatives $w \in W^J$. We denote by $\mathfrak{N}_J$ the image of $\mathfrak{N}$ under the canonical projection $p\colon Q_W \to Q_{W/W_J}$.

One of the major result concerning nil-Hecke rings says that the Chow ring $\mathrm{CH}_T(G/P_J)$ can be identified with the S-linear dual $\mathfrak{N}_J^\star = \mathrm{Hom}_S(\mathfrak{N}_J, S)$ (the parabolic dual), moreover, the pull-back map

$$\mathfrak{N}_J^\star \simeq \mathrm{CH}_T(G/P_J) \to \mathrm{CH}_T((G/P_J)^T) \simeq S^\star_{W/W_J}$$

induced by the T-fixed locus $(G/P_J)^T \hookrightarrow G/P_J$ is injective. This map is also called the moment map. Another observation that we are going to use is that the W-action on $\mathrm{CH}_T(G/P_J)$ can be explicitly translated into the action of $\mathfrak{N}$ (or S_W) on $\mathfrak{N}_J^\star$ as

$$p\delta_w \odot qf_v = pw(q)f_{wv},$$

here $\{f_v\}$ denotes the dual basis of $S_W^\star$ and $p, q \in S$. Using these observations one identifies the ring of W-equivariant endomorphisms with

$$End(\mathrm{CH}_T(G/P_J \times G/P_J)^W, \circ) \simeq End_{\mathfrak{N}\odot\text{-}}(\mathfrak{N}_J^\star).$$

Endomorphisms in terms of the Schubert basis.

We now describe W-equivariant endomorphisms of $\mathfrak{N}_J^\star$ in terms of the Schubert basis.

Recall that there is a free S-basis of $\mathrm{CH}_T(G/P_J)$ given by the so called Schubert classes ξ_w, $w \in W^J$. The latter are the classes of the Schubert varieties of dimensions $\ell(w)$ (the length of w), the closures of the orbits BwP_J/P_J in G/P_J. We have the following well-known formula for the $\odot$-action by divided difference operators (generators of $\mathfrak{N}$) on the Schubert classes

$$X_j \odot \xi_v = \begin{cases} \xi_{s_j v} & \text{if } l(s_j v) \geq l(v) \text{ and } s_j v \in W^J \\ 0 & \text{otherwise.} \end{cases}$$

Now let $\phi \in \mathrm{End}_{\mathfrak{N}}(\mathfrak{N}_J^\star)$. As an endomorphism of free S-modules ϕ is uniquely determined by its values on the classes ξ_w, $w \in W^J$ that is

$$\phi(\xi_w) = \sum_v a_{v,w} \xi_v,$$

where $(a_{w,v})$ is the corresponding generalized matrix of coefficients in S (here the coefficients are indexed by the product of left Bruhat posets in v and w). We may assume that all coefficients $a_{w,v}$ are homogeneous polynomials of degree $\ell(w) - \ell(v)$.

Since $\mathfrak{N}$ is generated by X_j's and elements of S, $\phi \in \mathrm{End}_{\mathfrak{N}\odot\text{-}}(\mathfrak{N}_J^\star)$ if and only if

$$X_j \odot \phi(\xi_w) = \phi(X_j \odot \xi_w) \text{ for all } j \text{ and } w \in W^J.$$

Using the properties of the $\odot$-action we compute the left hand side as follows (for $v, w \in W^J$)

$$\begin{aligned} X_j \odot \sum_v a_{v,w}\xi_v &= \sum_v (s_j(a_{v,w})X_j + X_j(a_{v,w})) \odot \xi_v \\ &= \sum_{s_j v \in W^J,\, l(s_j v) \geq l(v)} s_j(a_{v,w})\xi_{s_j v} + \sum_v X_j(a_{v,w})\xi_v \\ &= \sum_{v' \in W^J,\, l(s_j v') \leq l(v')} s_j(a_{s_j v',w})\xi_{v'} + \sum_v X_j(a_{v,w})\xi_v. \end{aligned}$$

As for the right hand side we obtain

$$\phi(X_j \odot \xi_w) = \begin{cases} \sum_v a_{v,s_j w}\xi_v & \text{if } l(s_j w) \geq l(w) \text{ and } s_j w \in W^J, \\ 0 & \text{otherwise.} \end{cases}$$

Combining, we obtain the following recursive formulas for the coefficients $a_{v,w}$:

Lemma 7.2. *If $l(s_j w) \geq l(w)$ and $s_j w \in W^J$, then*

$$a_{v,s_j w} = \begin{cases} s_j(a_{s_j v,w}) + X_j(a_{v,w}) & \textit{if } l(s_j v) \leq l(v), \\ X_j(a_{v,w}) & \textit{otherwise.} \end{cases}$$

If $l(s_j v) \leq l(v)$, then (observe that $s_j X_j = X_j$)

$$a_{s_j v,w} = \begin{cases} s_j(a_{v,s_j w}) - X_j(a_{v,w}) & \textit{if } l(s_j w) \geq l(w) \textit{ and } s_j w \in W^J, \\ -X_j(a_{v,w}) & \textit{otherwise.} \end{cases}$$

Corollary 7.3. *An endomorphism ϕ is uniquely determined by the W_J-invariant coefficients $a_v = a_{v,1} \in S^{W_J}$, $v \in W^J$ such that $l(s_j v) \geq l(v)$ for all $s_j \in W_J$.*

Motives of Severi-Brauer varieties

We fix a simple split group $G = \mathbb{G}_\Phi$ of type A_n and its standard maximal parabolic subgroup $P_J = \mathbb{P}_{\Phi,J}$, $J = \Delta \setminus \{1\}$. The respective projective homogeneous variety G/P_J is then isomorphic to the projective space $\mathbb{P}^n$. We consider the adjoint group $G = \mathrm{PGL}_{n+1}$ so that T^* is the root lattice with the basis given by simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. The Weyl group is given by $W = \langle s_1, \ldots, s_n \rangle$, where s_i is the simple reflection corresponding to α_i, $W_J = \langle s_2, s_3, \ldots, s_n \rangle$ and $W^J = \{1, v_1, v_2, \ldots, v_n\}$, where $v_i = s_i s_{i-1} \ldots s_1$.

Consider the nil-Hecke ring $\mathfrak{N}$ over the polynomial ring $S = \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$. Consider an idempotent ϕ and the associated generalized matrix of coefficients $(a_{v,w})$. For simplicity of notation set

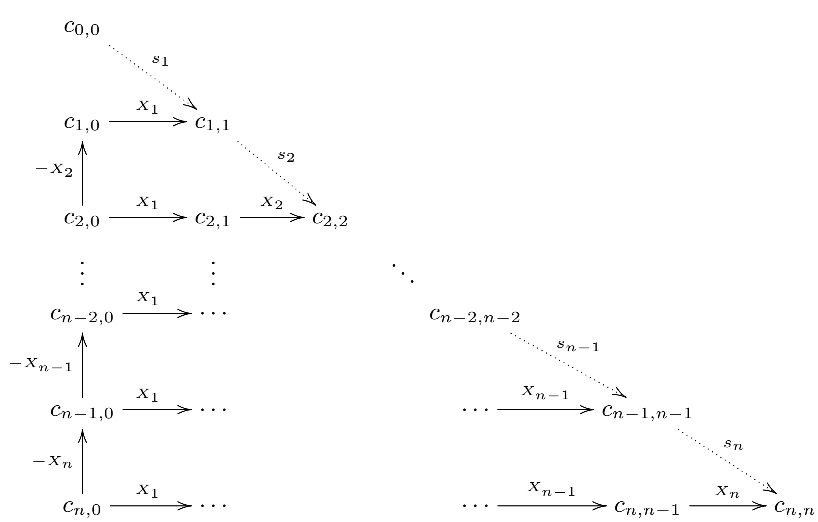
$$c_{i,j} := a_{v_i,v_j},\ c_{i,0} := a_{v_i,1},\ c_{0,j} = a_{1,v_j} \text{ for } i,j \geq 1, \text{ and } c_{0,0} = a_{1,1}.$$

So $(c_{i,j}) = (a_{v,w})$ is a lower-triangular idempotent $(n+1) \times (n+1)$-matrix in homogeneous polynomials $c_{i,j} \in \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$ of degrees $\deg c_{i,j} = i - j$.

The recursive formulas of Lemma **??** turn into:

$$\begin{aligned} c_{i,i} &= s_i(c_{i-1,i-1}) + X_i(c_{i,i-1}) && \text{if } i \geq 1, \\ c_{i,j} &= X_j(c_{i,j-1}) && \text{if } i \neq j,\ j \geq 1, \\ c_{i,0} &= (-1)^{n-i} X_{i+1,..,n}(c_{n,0}) && \text{for } n > i \geq 1. \end{aligned}$$

which can be expressed by the following diagram of operators



These formulas imply (over $\mathbb{Z}$):

$$c_{i,j} = X_{j,..,1}(c_{i,0}) = (-1)^{n-i} X_{j,..,1,i+1,..,n}(c_{n,0}) \text{ for } i > j \geq 1 \quad \text{(under the diagonal)},$$
$$c_{i,i} = s_i(c_{i-1,i-1}) + (-1)^{n-i} X_{i,..,1,i+1,..,n}(c_{n,0}) \quad \text{(on the diagonal)}.$$

In particular, the matrix $(c_{i,j})$ and, hence, the idempotent ϕ is uniquely determined by the homogeneous polynomial $c_{n,0}$ of degree n.

Using relations in the nil Hecke ring $\mathfrak{N}$ for the divided difference operators X_j and the fact that $X_j(c_{n,0}) = 0$ for all $j \neq 1, n$ (Corollary **??**) we get

$$X_k \circ X_{i-1,..,1,i+1,..,n}(c_{n,0}) = 0, \quad \text{for all } k \neq i.$$

So $c_{i,i-1} \in S^{W_i}$, where W_i is generated by all simple reflections except the i-th one. Since $c_{i,i-1}$ has degree 1, we can express it as $c_{i,i-1} = b_1\alpha_1 + \ldots + b_n\alpha_n$, $b_i \in \mathbb{Z}$. Then $c_{i,i-1} \in S^{W_i}$ is equivalent to

$$b_2 = 2b_1, b_3 = 3b_1, \ldots, b_i = ib_1 = (n+1-i)b_n, \ldots, b_{n-2} = 3b_n, b_{n-1} = 2b_n.$$

Assume $n+1 = p^r$ for some prime p and $r \geq 1$. Then

$$p \mid X_i(c_{i,i-1}) = 2b_i - b_{i-1} - b_{i+1} = b_1 + b_n.$$

Since ϕ is an idempotent, all diagonal elements $c_{i,i}$ have to be idempotent as well, i.e. each $c_{i,i}$ is either 0 or 1. The recursive formulas and the fact that $p \mid X_i(c_{i,i-1})$ then imply that $c_{i,i} = c_{i-1,i-1}$ for all i, i.e. that there are no nontrivial idempotents. In other words, we obtain

Proposition 7.4. *Let $G = \mathrm{PGL}_{n+1}$, where $n = p^r - 1$ for some $r \geq 1$ and a prime p. Let P_J be the standard parabolic subgroup of type A_{n-1}.*

Then the $\mathfrak{N}$-module $\mathfrak{N}_J^\star$ is indecomposable over $\mathbb{Z}$.

Remark 7.5. This fact implies the celebrated result by Karpenko on indecomposability of the integral motive of the Severi-Brauer variety of a generic algebra.

Exercise 7.6. Assuming the recursive formulas work out the case of $n = 3$ step by step.

Exercise 7.7. This is not necessary for the class but... just for fun.

Consider the semisimple group $G = \mathrm{SL}_2 \times \ldots \times \mathrm{SL}_2 / \Delta(\mu_2)$ (product of simple groups SL_2 modulo the central diagonal subgroup μ_2). Take $P = B$ the Borel subgroup, so the variety G/P_J is the product $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$. What one can say about $End_{\mathfrak{N}\odot\text{-}}(\mathfrak{N}_J^\star)$: do there exist non-trivial idempotents? If yes, how many and how to describe them?

References

[CNZ] Calmès, B., Neshitov, A., Zainoulline, K. Relative equivariant motives versus modules. Canadian Journal of Math. 73 (2021), Issue 1, 131-159.